Practice with Parameterizing Curves and Surfaces

Part 2: Surfaces

To calculate (in fact, to define) the surface area of a 2-dimensional surface in 3-space, and to calculate (in fact, to define) an integral whose domain is some surface, we need to parameterize the surface. Since the surface is a 2-dimensional thing, we expect to need 2 parameters; since the surface lives in 3-space, we expect to have 3 coordinate functions.

So a parameterization of a surface in 3-space is a function from some subset of $\mathbb{R}^2$ into $\mathbb{R}^3$.

We can denote such a function as $X(s,t) = [x(s,t), y(s,t), z(s,t)]$.

Example 1.

Parameters $(s,t)$
Coordinate functions $x(s,t) = s + t^2$
$y(s,t) = \cos(s)$
$z(s,t) = \sin(t)$

Let's consider two different domains:
Example 1a. $s=0..1, t=0..1$ a square in $(s,t)$ space.
Example 1b. $s=-1..1, t=-\sqrt{1-s^2} .. \sqrt{1-s^2}$ a circular disk in $(s,t)$ space.

```maple
> with(plots):
Warning, the name changecoords has been redefined
> X := (s, t) -> [s+t^2, cos(s), sin(t)];
X := (s, t) -> [s + t^2, cos(s), sin(t)]
> plot3d(X(s,t), s=0..1, t=0..1);`
The surface $S$ may be easier to see if we make the picture prettier.

> plot3d(X(s,t), s=0..1, t=0..1, axes=boxed, lightmodel=light1, labels=[x,y,z], color=pink, orientation=[115,73]);
Now let's use the same function $X(s,t)$, but apply it to a different domain in $(s,t)$ space.

\[ \texttt{plot3d}(X(s,t), \ s=-1..1, \ t=\text{-sqrt}(1-s^2)..\text{sqrt}(1-s^2)); \]
It will be much easier to visualize the surface if you use the computer to do the plot, then play with the picture by rotating or zooming. But we can make a static picture a bit more informative.

> plot3d(X(s,t), s=-1..1, t=-sqrt(1-s^2)..sqrt(1-s^2), color=yellow, labels=[x,y,z], axes=boxed, lightmodel=light1, orientation=[42,39]);
Examples with more familiar surfaces

Example 2.  S = a sphere, or part of a sphere

Here are some parameterizations of a standard sphere: radius = 3, center = [0,0,0], or parts of that sphere.

Solution 2a. Use "spherical coordinates"

\[ x = \rho \sin(\phi) \cos(\theta) \]
\[ y = \rho \sin(\phi) \sin(\theta) \]
\[ z = \rho \cos(\phi) \]

For a sphere of fixed radius 3, we just use \( \rho = 3 \). The two parameters are the angles \( \phi \) and \( \theta \).

> plot3d([3*sin(phi)*cos(theta), 3*sin(phi)*sin(theta), 3*cos(phi)],
> phi=0..Pi, theta=0..2*Pi, scaling=constrained);
To parameterize the upper hemisphere, we just restrict phi to run from 0 to Pi/2.

```maple
> plot3d([3*sin(phi)*cos(theta), 3*sin(phi)*sin(theta), 3*cos(phi)],
         phi=0..Pi/2, theta=0..2*Pi, scaling=constrained, orientation=[51,104]);
```
To parameterize the right hemisphere (i.e. x coordinates positive), use phi=0..Pi, but have theta to run between -pi/2 and pi/2.

> plot3d([3*sin(phi)*cos(theta), 3*sin(phi)*sin(theta), 3*cos(phi)],
  phi=0..Pi, theta=-Pi/2..Pi/2, scaling=constrained, orientation=[51,104],
  orientation=[-116,70], axes=boxed, labels=[x,y,z]);
**Exercise:** Parameterize the portion of the sphere of radius 5 centered at the origin in $\mathbb{R}^3$ that lies in the first octant.

**Exercise:** Parameterize the portion of the sphere of radius 5 centered at the origin in $\mathbb{R}^3$ that lies in the octant where $y$ is negative, $x$ is negative, and $z$ is positive.

What about the portion of the sphere of radius 3 that lies above some $z$-value (think of a polar ice cap). We want to restrict, e.g., $z > 2$.

In spherical coordinates, $z = \rho \cos(\phi) = [\text{in our case}] 3 \cos(\phi)$.

So we want to restrict the angle $\phi$ to make $3 \cos(\phi) > 2$. That says $\cos(\phi) > 2/3$, so we need to have $\phi$ run from 0 to $\arccos(2/3)$.

```plaintext
> plot3d([3*sin(phi)*cos(theta), 3*sin(phi)*sin(theta), 3*cos(phi)], phi=0..arccos(2/3), theta=0..2*Pi, scaling=constrained, orientation=[104,96], axes=boxed, labels=[x,y,z]);
```
Example 3.

Parameterize a cylinder, or part of a cylinder.

For example, parameterize a circular cylinder (i.e. cross-section is a circle) in $\mathbb{R}^3$ of radius 3 and height 5, whose axis is the y-axis. [I have to say "a" rather than "the" cylinder because I have not specified where the base is located. Let's say the base is supposed to lie in the plane $y=1$.]

We need two parameters: To describe a cylinder we might expect to use cylindrical coordinates, something like $(r,\theta,z)$. But not all three coordinates are independent here; and the "z" coordinate that is supposed to correspond to "height" is supposed to run along the y-axis here, not the z-axis. This can get confusing. So we will use $(s,t)$ instead of suggestive letters like "r" or "theta".

$s$ is the polar angle, running from 0 to $2\pi$

$t$ is the height, running [we are instructed] from 1 to $1+5=6$. 
\[ X(s,t) = [x,y,z], \text{ where} \]
\[ x(s,t) = 3 \cos(s) \]
\[ y(s,t) = t \]
\[ z(s,t) = 3 \sin(s) \]

> `plot3d([3*cos(s), t, 3*sin(s)], s=0..2*Pi, t=1..6);`

Let's make the picture better, and include the axes so we can see we have the right cylinder.

> `plot3d([3*cos(s), t, 3*sin(s)], s=0..2*Pi, t=1..6, scaling=constrained, axes=normal, labels=[x,y,z], view=[-6..6, -3..12, -6..6]);`
**Exercise:** Parameterize a cylinder of radius 2 and length 4 whose axis is the x-axis.

**Exercise:** Parameterize a cylinder whose cross-section is an ellipse with radii 2 and 3, and whose axis is the x-axis.

**Exercise (for anyone who is finding these too easy) Parameterize a cylinder whose cross-section is a circle of radius 1, whose length is 5, and whose axis is the line y=x in the xy-plane.** Here is a picture, just to prove it can be done.

>  

>