Practice with Surface Integrals

In each of the following examples, we will first parameterize a surface, then set up (maybe evaluate) an iterated integral representing the surface area; then the integral of some scalar function (think of "density") defined on the surface; then the flux of one or more vector fields across (i.e. through) the surface.

When I tell the computer to draw the surface, I will do that in two stages: Step 1 = just parameterize the surface; Step 2 = display the plotted surface in a way to make the picture as clear as I can, so including various plotting options.

> with(plots): with(linealg):

Note: We will have several occasions to calculate the length of a given vector. Maple's built-in routine for this is a little awkward, so let's define the norm function ourselves; likewise for the dot product of two vectors.


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Example 1.
Surface \( S \) = the portion of the paraboloid \( z=9-x^2-y^2 \) for which \( z \) is nonnegative.

Scalar function \( f(x,y,z) = \cos(xy-z) \)

Vector field \( F(x,y,z) = 5 \mathbf{i} \) or vector field \( G(x,y,z) = x \mathbf{i} + y \mathbf{j} + (z-x) \mathbf{k} \)

> X := [s,t,9-s^2-t^2];

\[
X := [s, t, 9-s^2-t^2]
\]

> DrawSurface := plot3d([s,t,9-s^2-t^2], s=-3..3, t=-sqrt(9-s^2) ..sqrt(9-s^2));

> display(DrawSurface, axes=normal, labels=[x,y,z], scaling=constrained, view=[-5.5, -5..5, -2..11], labelfont=[HELVETICA, ROMAN, 18], orientation=[46,106], lightmodel=light1);
Remark on the sign of vector \( \mathbf{N} \). For calculating surface area, we can use \( \mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t \) OR \( \mathbf{N} = \mathbf{T}_t \times \mathbf{T}_s \), since ultimately we are planning to take the magnitude of this vector, so the choice of this direction vs. that direction does not matter. But when we are calculating flux, then flipping \( \mathbf{N} \) will produce the negative of what we'd get with the other \( \mathbf{N} \). So just be careful about making the person who asks the question tell you which direction of flow through the surface is considered "positive".

We calculated \( \mathbf{T}_s \times \mathbf{T}_t \) above by using Maple's built-in operator "crossprod". To do this "by hand", calculate the cross product in the usual way, as a 3x3 determinant.

(remark: The operator "stackmatrix" is used below to construct a matrix by stacking the vectors one above another.)
> M:=stackmatrix([i,j,k], Ts, Tt);

\[
M:=
\begin{bmatrix}
i & j & k \\
1 & 0 & -2s \\
0 & 1 & -2t \\
\end{bmatrix}
\] (1)

> AlsoN:=det(M);

\[AlsoN := 2is+2jt+k\]

> JacStretch:=NormVec(N);

\[\text{JacStretch} := \sqrt{4s^2+4t^2+1}\]

> SurfaceArea:=Int(Int(JacStretch, \ t=-sqrt(9-s^2)..sqrt(9-s^2)), \ s=-3..3);

\[
\text{SurfaceArea} := \int_{-3}^{3} \sqrt{9-s^2} \sqrt{4s^2+4t^2+1} \ dt \ ds
\]

(I will let you decide if you have the machinery to evaluate this integral.)

The integral of scalar function \(f(x,y,z)\) on this surface is calculated as follows.

> f:=(x,y,z) \rightarrow \cos(x*y-z);

\[f := (x, y, z) \rightarrow \cos(yx-z)\]

> f_in_terms_of_st:=f(X[1], X[2], X[3]);

\[f_{ \text{in terms of st}} := \cos(ts-9+s^2+t^2)\]

Then

> IntOf_f:=Int(Int(f_in_terms_of_st * JacStretch, \ t=sqrt(9-s^2)..sqrt(9-x^2)), \ s=-3..3);

\[
\text{IntOf}_f := \int_{-3}^{3} \cos(ts-9+s^2+t^2) \sqrt{4s^2+4t^2+1} \ dt \ ds
\]

The (flux) integral of a vector field across the surface is the amount of "flow" that gets through the surface in a unit of time. We will work with the two given sample vector fields

> Field_F:=[5,0,0];

\[\text{Field}_F := [5, 0, 0]\]

> Field_G:=[x,y,z-x];

\[\text{Field}_G := [x, y, z-x]\]

> DrawField_F:=fieldplot3d(Field_F, x=-5..5, y=-5..5, z=0..10, arrows=SLIM, grid=[5,5,5], color=blue):

> DrawField_G:=fieldplot3d(Field_G, x=-4..4, y=-4..4, z=0..10, arrows=SLIM, grid=[5,5,5], color=blue):

> display({DrawSurface, DrawField_F});
> display({DrawSurface, DrawField_G});
It looks intuitively as if the net flow of field F through the surface is zero, with as much coming in as going out. On the other hand, the picture of field G suggests there is a net flow "up" through the surface.

To calculate the flux, we first compute the flux integrand, (vector field) dot (normal vector N), then integrate that as a function of the parameters (s,t).

For vector field F....
> Field_F;

[5, 0, 0]

> N;

N

(Sometimes Maple does not display a variable as what it denotes, but just shows the name. This is annoying, but we can push the computer to say what N is...)
> evalm(N);

[2 s, 2 t, 1]

> FluxIntegrand:=DotProd(Field_F,N);
\[ \text{FluxIntegrand} := 10 \text{ s} \]

> Flux:=Int(Int(FluxIntegrand, t=-sqrt(9-s^2) .. sqrt(9-s^2)), s=-3 .. 3) ;

\[ Flux := \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} 10 \text{ s} \, dt \, ds \]

> value(Flux);

0

As our intuition suggested, the flux of the vector field \( F \) through this surface is 0.

Now let's see what happens with field \( G \).

> Field_G;

\[ [x, y, z-x] \]

> evalm(N);

\[ \begin{bmatrix} 2s & 2t & 1 \end{bmatrix} \]

> FluxIntegrand:=DotProd(Field_G,N);

\[ \text{FluxIntegrand} := 2x s + 2y t + z-x \]

> Flux:=Int(Int(FluxIntegrand, t=-sqrt(9-s^2) .. sqrt(9-s^2)), s=-3 .. 3) ;

\[ Flux := \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} (2x s + 2y t + z-x) \, dt \, ds \]

This integral cannot be evaluated yet. At each point on the surface, we have the normal vector \( N \) expressed in terms of parameters \( (s,t) \); but the vector field \( G \) is expressed in terms of the position \( (x,y,z) \). We need to write the vector field, for points on the surface, in terms of the parameters \( (s,t) \).

> X;

\[ [s, t, 9-s^2-t^2] \]

> G_along_the_surface:=subs(x=X[1], y=X[2], z=X[3], Field_G);

\[ G_{\text{along the surface}} := [s, t, 9-s^2-t^2] \]

Now the flux integrand \( (G \text{ dot } N) \) is expressed in terms of \( (s,t) \)...

> FluxIntegrand:=DotProd(G_along_the_surface,N);

\[ \text{FluxIntegrand} := s^2 + t^2 + 9-s \]

> Flux:=Int(Int(FluxIntegrand, t=-sqrt(9-s^2) .. sqrt(9-s^2)), s=-3 .. 3) ;

\[ Flux := \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} (s^2 + t^2 + 9-s) \, dt \, ds \]

> value(%);
\[ \frac{243}{2} \pi \]

I have no pretty/intuitive interpretation for this number - it's just the flux of the given vector field \( \mathbf{G} \) through the surface. But we can at least see if it is reasonable to have the flux be positive.

The field \( \mathbf{G} \) seems to be flowing up and out through the surface. How does the vector \( \mathbf{N} \) lie relative to the surface? It is a *normal* vector, so it is perpendicular to the surface; but which way does it point?

Let's experiment near some point on the surface, e.g. where \( s=1.7 \) and \( t=0.93 \) (I'm trying to pick "random" points)

\[
\begin{align*}
N_{\text{local}} & := \text{subs}(s=1.7, t=0.93, [N[1], N[2], N[3]]) ; \\
& N_{\text{local}} := [3.4, 1.86, 1] \\
X_{\text{local}} & := \text{subs}(s=1.7, t=0.93, X) ; \\
& X_{\text{local}} := [1.7, 0.93, 5.2451] \\
\text{TubeToDraw} & := \text{evalm}(X_{\text{local}} + a\times N_{\text{local}}) ; \\
& \text{TubeToDraw} := [1.7 + 3.4 a, 0.93 + 1.86 a, 5.2451 + a] \\
\text{DrawN} & := \text{tubeplot}(\text{TubeToDraw}, a=0..1, \text{scaling=constrained}, \text{radius=0.1}, \text{color=red}, \text{style=patchnogrid}) ; \\
\text{Draw\_G\_locally} & := \text{fieldplot3d}(\text{Field\_G}, x=1..3, y=0..2, z=4..6, \text{arrows=SLIM}, \text{grid=[3,3,3]}, \text{color=blue}) ; \\
\text{G\_at\_that\_point} & := \text{subs}(x=X_{\text{local}}[1], y=X_{\text{local}}[2], z=X_{\text{local}}[3], \text{Field\_G}) ; \\
& \text{G\_at\_that\_point} := [1.7, 0.93, 3.5451] \\
\text{OneVectorG} & := \text{evalm}(X_{\text{local}} + b\times \text{G\_at\_that\_point}) ; \\
& \text{OneVectorG} := [1.7 + 1.7 b, 0.93 + 0.93 b, 5.2451 + 3.5451 b] \\
\text{DrawOneVectorG} & := \text{tubeplot}(\text{OneVectorG}, b=0..1, \text{scaling=constrained}, \text{radius=0.1}, \text{style=patchnogrid}, \text{color=blue}) ; \\
\text{display} & \{\text{DrawSurface}, \text{DrawN}, \text{DrawField\_G}, \text{DrawOneVectorG}\}, \text{orientation=[-33,104]} ;
\end{align*}
\]
At least we can see that at a "typical" point on the surface, the field $\mathbf{G}$ has a positive component in the direction of $\mathbf{N}$. 