GEOMETRY OF CURVES

Text reading guide. Study Section 3.2, pages 190–197, 201-202 (you may also find the summary boxes on page 203 –204 helpful. (You can skip anything with a B and the cute formula for κ at the end of the final box.) The material in Section 3.2, including the Frenet-Serret formulas in the Adendum p. 204 –206, is near and dear to my heart, some relating to my own research interests. But it’s just a little too much for us to cover.

The text studies curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. We will be focusing on curves just in $\mathbb{R}^2$.

Geometric properties of a curve. (We are assuming for this whole discussion that we have a smooth curve.) A curve in space is a “thing”. It has geometric properties, in particular length and curvature. The length of the curve measures how much “stuff” we need to make the object. If the object is a bent springy wire, then the curvature measures how hard it is to bend the wire to make the curve. If the object is a highway, then the curvature measures how hard it is for a car to make the turn. Your own driving experience tells you that the curvature is not the only factor in deciding how hard it is to make some turn; we also have to pay attention to how fast the car is going. (See the later section on Acceleration).

We can measure these physical properties by parameterizing the curve as some function $x \rightarrow x(t)$ and using the vector-valued function $x(t)$ to calculate the geometric properties. This is an application of our study of vector-valued functions. Also, the particular vectors $T$ and $N$ defined below will be used later in the course when we study the idea of flux.

Consider the curve in Figure 1. We can parameterize this curve as

$$x(t) = (t \cos t)i + (t \sin t)j \quad (t = 0..6).$$
There are 8 points marked along the curve; these do NOT correspond to equal steps in the time parameter $t$. Starting from the origin (corresponding to $t = 0$), I have marked points along the curve measuring distance along the curve: the 8 points are at distances 0, 2, 4, 6, 8, 10, 12, 14 from the beginning. (It is not obvious how to calculate which times $t$ produce these distance steps; more about this later.)

**Calculating arclength.** If $x(t)$ gives position at time $t$, then, as we have discussed in class, the derivative $x'(t)$ is the velocity. Here we have

$$v(t) = (\cos t - t \sin t)i + (\sin t + t \cos t)j.$$  

The speed is the magnitude of the velocity. Here we have

$$||v(t)|| = \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} = \sqrt{t^2 + 1}.$$

The length along the curve, from $t = 0$ to $t = 6$ is the integral of speed:

$$L = \int_{t=0}^{6} \sqrt{t^2 + 1} \, dt = 3\sqrt{37} - 1/2 \ln(-6 + \sqrt{37}) \approx 19.5.$$

These arclength integrals tend to be difficult to evaluate. In this example, Maple (or a student who remembers trig substitution from Calc II) is able to evaluate the integral exactly. Except for very special cases (such as you will have in homework problems), often the only route is to numerically approximate the integrals.

To find the values of $t$ that produce arclengths $= 1$, $2$, etc., I had Maple numerically solve for the times. (If anyone wants to learn more about how to do this, I will be happy to help.) The times worked out to

0 1.527853327 2.428806853 3.108090381
3.673763864 4.168011462 4.612194765 5.018838531

If you calculate the differences between these time values, you will see that the differences are getting smaller. As we go farther along the curve, it takes less time to cover one unit of arclength. That is consistent with the formula we have for speed: $\sqrt{t^2 + 1}$. The speed increases as $t$ increases.

**The arclength parameter.** Introduce a new variable, $s$, representing distance along the curve. We can view $s$ as a function of $t$,

$$s(t) = \int_{\tau=0}^{t} ||v(\tau)|| \, d\tau.$$  

(The text uses dummy variable $\tau$ for this integral, so I have also used $\tau$.)

In our spiral example, we have

$$s(t) = \int_{\tau=0}^{t} \sqrt{\tau^2 + 1} \, d\tau = (1/2)\sqrt{t^2 + 1} + (1/2) \ln(t + \sqrt{t^2 + 1}).$$  

This is pretty messy, but it still makes sense. The function $s(t)$ may be hard to calculate, but its derivative $\frac{ds}{dt}$ is easy: it is just the speed of motion along the curve. From the calc I “Fundamental Theorem of Calculus”, if $s(t)$ is defined as the above integral, then its derivative is the integrand:

$$\frac{ds}{dt} = ||v(t)||.$$  

We also can think of $s$ as the *independent* variable, with the coordinates of points on the curve viewed as functions of how far the points are along the curve. In order to express $t$ in terms of $s$, in the above example, we would have to solve the equation

$$s = (1/2)t\sqrt{t^2 + 1} + (1/2)\ln(t + \sqrt{t^2 + 1})$$

for $t$. We do not want to try to do this explicitly!! But, in theory, so long as the speed along a curve is continuous and never 0, then $t$ is in fact a differentiable function of $s$. Usually, we will do our calculations in terms of $t$, but sometimes make definitions in terms of $s$.

The unit tangent vector $T$. The velocity vector $v(t)$ is tangent to the curve and its length is the speed. If we scalar multiply the velocity by $(1/$speed$)$, we get a unit vector $T$ tangent to the curve. We can view $T$ as a function of time $t$ or as a function of position $s$ along the curve.

$$T(t) = \frac{v(t)}{∥v(t)∥}.$$ 

In our spiral curve example, we have

$$v(t) = (\cos t - t \sin t)i + (\sin t + t \cos t)j$$

and

$$T(t) = \frac{\cos t - t \sin t}{\sqrt{t^2 + 1}}i + \frac{\sin t + t \cos t}{\sqrt{t^2 + 1}}j$$

In Figure 2, we see the vectors $T(t)$ drawn at several points along the curve (the same points as used earlier, spaced equally in arclength along the curve). 

![Figure 2. Curve with unit tangent vectors $T$ at several points](image-url)
**The derivative of T.** As we move along the curve, the unit tangent vector $T$ changes direction, but it does not change length. We have a lemma from the previous section saying that if a vector function does not change length, then the derivative of that vector function is perpendicular to the vector. So if we draw the derivative $\frac{dT}{dt}$ at a point on the curve, that derivative vector will be perpendicular to $T$. We show in Figure 3 the normal vectors $\frac{dT}{dt}$ at two points. The (blue) normal vectors are not the same length, approx. 1.3 compared to 1.04. (The inner one is longer).

![Figure 3. Curve with normal vectors $\frac{dT}{dt}$ at two points](image)

We have a formula for $T(t)$ above, and you can calculate $\frac{dT}{dt}$ directly from that, even if it gets messy.

$$\frac{dT}{dt} = -\frac{t^2 + 2}{(t^2 + 1)^{3/2}}((\sin t + t \cos t)\mathbf{i} + (-\cos t + t \sin t)\mathbf{j}).$$

Let’s ignore the messy formulas and just look at the geometry. The derivatives $\frac{dT}{dt}$ measure how the unit vectors $T$ change as we move along the curve. The lengths don’t change, but the directions do. So $\frac{dT}{dt}$ measure how fast the unit tangent vector $T$ turns (with respect to time) as we move along the curve. This almost captures the idea that the curve seems “tighter” or “more curvy” in some places than others. However, the time derivative $\frac{dT}{dt}$ does not depend just on the shape of the curve, but also on the particular parameterization we are using. If we traversed the same curve twice as fast, then $\frac{dT}{dt}$ would be twice as big.

In our example above, the curve seems much tighter at the first point than at the second; but in the given parameterization, we are traveling faster at the second point than at the first, so this exaggerates the curvature at the second point and makes the two values for $\frac{dT}{dt}$ closer to each other than the shape “wants” them to be.

In order to define a geometric notion of curvature, that depends just on the shape of the curve, and not on the particular parameterization, we want to study how the unit tangent changes with respect to distance moved along the curve, not with respect to time. With this in mind, we define the curvature

$$\kappa = \left\| \frac{dT}{ds} \right\|.$$  

By the “chainrule”, we have

$$\frac{dT}{dt} = \frac{dT}{ds} \frac{ds}{dt},$$

so we can also define curvature as

$$\kappa = \left\| \frac{dT}{ds}/\text{speed} \right\| = \left\| \frac{dT}{dt}/\|v\| \right\|.$$
For the given curve, we have calculated $\frac{dT}{dt}$ above; we can find $\frac{dT}{ds}$ just by dividing through by (speed), and then $\kappa$ is the length of that vector.

$$\frac{dT}{dt} = -\frac{t^2 + 2}{(t^2 + 1)^2} \left( (\sin t + t \cos t) i + (-\cos t + t \sin t) j \right).$$

and (after simplifying all the sines and cosines)

$$\kappa = \frac{t^4 + 4t^2 + 4}{(t^2 + 1)^3}.$$ 

At the two points shown in Figure 3, the values for $\kappa$ are approximately 0.51 and 0.04; the curvature at the first point is around 12 times larger than the curvature at the second point. In Figure 4, we see two curvature vectors $\frac{dT}{ds}$, one around 6.4 times larger than the other. The number $\kappa$ measures the geometric curvature at different points of the curve. (In order to make the two curvature vectors visible, but keep their relative lengths the same, I have scaled each of them to 5 times their actual lengths; and I didn’t even try to plot the curvature at the outer point, because the arrow would be so small.)

\[ \text{Figure 4. Curve with normal vectors } \frac{dT}{dt} \text{ at two points} \]

**The unit normal $N$.** Just as it was useful to normalize $v$ to define the unit tangent $T$, it also is useful to define a **unit normal vector** $N$.

$$N = \frac{dT}{\|dT/dt\|} = \frac{dT}{\|dT/ds\|} = \frac{dT}{\kappa}.$$ 

Equivalently,

$$\frac{dT}{ds} = \kappa N.$$ 

The vector $\frac{dT}{ds}$ is perpendicular to $T$, points in the direction that the curve is turning (notice this would be the same even if we reversed the direction of traversing the curve), and has length that measures how tightly the curve is turning at a given point.

**Tangential and normal components of acceleration.** If we see an object moving along a certain curve (the speed might be changing), the force causing the movement (equivalent, up to a multiplicative constant to the acceleration of the object) can be resolved into one component that pushes along the curve, and one component that pushes perpendicular to the curve. The tangential push affects the speed but not the direction of movement; the normal push affects the direction but not the speed. Formally, here is how to resolve acceleration into tangential and normal components.
\[ \mathbf{v} = \frac{d\mathbf{s}}{dt} \mathbf{T} \implies \frac{d\mathbf{v}}{dt} = \left( \frac{d}{dt} \frac{d\mathbf{s}}{dt} \right) \mathbf{T} + \frac{d\mathbf{s}}{dt} \left( \frac{d\mathbf{T}}{dt} \right) \]
\[ = \left( \frac{d^2\mathbf{s}}{dt^2} \right) \mathbf{T} + \left( \frac{d\mathbf{s}}{dt} \right)^2 \frac{d\mathbf{T}}{ds} \]
\[ \mathbf{a} = \left( \frac{d^2\mathbf{s}}{dt^2} \right) \mathbf{T} + \left( \frac{d\mathbf{s}}{dt} \right)^2 \kappa \mathbf{N}. \]

This formula says something obvious, and something perhaps surprising. The “obvious” part is that the tangential component of acceleration equals the acceleration along the curve: pushing in the direction of motion changes the speed in proportion to how hard is the pushing. The less obvious part is the normal component of acceleration. It has two terms: one involving the shape of the curve (the curvature \( \kappa \)) and the other the square of the speed. This is why, when you drive through a curve at high speed, there is a great tendency to skid: the force normal to the curve (that has to be overcome by friction between the tires and the road) is proportional to how tight is the curve (\( \kappa \)) and the square of how fast you are driving. If you double your speed, you multiply the normal component of acceleration by 4.

**Practical calculations.** If someone asks you to resolve the acceleration of some given \( \mathbf{x}(t) \) into tangential and normal components, you need to find \( \frac{d^2\mathbf{s}}{dt^2} \) and the product \( \left( \frac{d\mathbf{s}}{dt} \right)^2 \kappa \). Some of the calculations are straightforward (even if sometimes long), others even harder. To find \( \frac{d^2\mathbf{s}}{dt^2} \), calculate the velocity vector \( \mathbf{v}(t) \), then the speed \( \| \mathbf{v}(t) \| \), and then the derivative of that length. You also should calculate the acceleration vector \( \mathbf{a} \) directly, as the derivative of \( \mathbf{v} \), in order to subsequently find \( \kappa \), using the method below. (You can calculate \( \kappa \) directly, but it is messier.) There is a trick that sometimes makes finding \( \kappa \) a little easier.

Since the vectors \( \mathbf{T} \) and \( \mathbf{N} \) are orthogonal unit vectors, the length of any vector \( a\mathbf{T} + b\mathbf{N} \) is just \( \sqrt{a^2 + b^2} \). (Proof: Take the dot product of the vector with itself; that gives the square of the length; use the facts that \( \mathbf{T} \) and \( \mathbf{N} \) are orthogonal and unit vectors to simplify what you got to just \( (a^2 + b^2) \).) Once you know \( \|a\| \) and \( \frac{d^2\mathbf{s}}{dt^2} \), you can use the equation

\[ \|a\|^2 = \left( \frac{d^2\mathbf{s}}{dt^2} \right)^2 + \left( \frac{d\mathbf{s}}{dt} \right)^2 \kappa^2 \]

\[ \text{to find } \left( \frac{d\mathbf{s}}{dt} \right)^2 \kappa. \]

If you want to find just \( \kappa \), perhaps the easiest way is with the formula (on page 203) that uses the cross product. Treat the vectors as 3-dimensional; just view \( a\mathbf{i} + b\mathbf{j} \) in \( \mathbb{R}^2 \) as \( a\mathbf{i} + b\mathbf{j} + 0\mathbf{k} \) in \( \mathbb{R}^3 \).

\[ \kappa = \frac{\| \mathbf{v} \times \mathbf{a} \|}{\| \mathbf{v} \|^3}. \]

(end of handout)