Finding local maxima and minima for functions $z=f(x,y)$

[the cookbook]

> `with(plots):`

I want very much for you to understand all the math that goes in to the process of finding points $(x_0, y_0)$ where a function $z=f(x,y)$ might have a local max or local min, and the process of using the Hessian matrix to try to decide if a given (smooth) critical point is indeed a local max or min or a saddle point. HOWEVER, I also know that students sometimes want to have a recipe to follow for complicated problems. So, here's the recipe that will work for most of your homework and exam problems on this topic.

1. Calculate the partial derivatives $df/dx$ and $df/dy$.
2. Form a system of 2 equations in 2 unknowns by setting $df/dx = 0$ and $df/dy = 0$.
3. Solve the system to find all points $(x_0, y_0)$ at which both partial derivatives are 0.
4. Calculate the second derivatives $f_{xx}$, $f_{xy} (=f_{yx})$, and $f_{yy}$.
5. Form the Hessian matrix $[[f_{xx}, f_{xy}], [f_{yx}, f_{yy}]]$
   - Remark: I know $f_{xy} = f_{yx}$ for the functions we are likely to encounter in class; but writing $f_{xy}$ once and $f_{yx}$ the second time is a nice memory aide to help you learn what the Hessian looks like for functions of 3 or more variables. It is the *derivative* matrix of the vector function grad($f$).
6. Test each critical point according to the following "differential diagnosis" (borrowing that term from medicine, not from math):
   - Hessian determinant $= 0$ $\Rightarrow$ second derivative test for max vs. min vs saddle fails; try perturbing $(x_0, y_0)$ to see the induced changes in $f(x,y)$.
   - Hessian determinant negative $\Rightarrow$ definitely a saddle point
   - Hessian determinant positive $\Rightarrow$ definitely a local max or local min; consult $f_{xx}$ (or, equivalently, consult $f_{yy}$) to see which.

Here are some examples:

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> `f:=x*y + 8/x + 1/y;`

\[ f := xy + \frac{8}{x} + \frac{1}{y} \]  \hspace{1cm} (1)

> `fx:=diff(f,x);`

\[ fx := y - \frac{8}{x^2} \]  \hspace{1cm} (2)

> `fy:=diff(f,y);`
\[ fy := x - \frac{1}{y^2} \]  
(3)

\[
\text{SystemOfEquations} := \{fx=0, fy=0\};
\]

\[
\text{SystemOfEquations} := \left\{ y - \frac{8}{x^2} = 0, x - \frac{1}{y^2} = 0 \right\}
\]  
(4)

The system is not linear, so I recommend always using substitution to try to solve. Here the first equation says \( y = 8/x^2 \). Plug this into the second equation to make that \( x = 1/(8/x^2)^2 \), i.e. \( x = x^4/64 \). Since x cannot be 0 (from the first equation), divide both sides by \( x \) to get \( 1 = x^3/64 \), which says \( x = 4 \). Go back to the \( y = 8/x^2 \) equation to see that if \( x = 4 \) then \( y = 8/16 = 1/2 \). So \( (4, 1/2) \) is the only critical point.

Before going on, let's ask the computer to check our solution:

\[
\text{solve(SystemOfEquations, \{x,y\});}
\]

\[
\{ y = \frac{1}{2}, x = 4 \}, \{ y = \frac{1}{2} \text{ RootOf} \left( Z^2 + Z + 1, \text{label} = \_L1 \right), x = 4 \text{ RootOf} \left( Z^2 + Z + 1, \text{label} = \_L1 \right) \}
\]  
(5)

The only solution in real numbers is the one we found. The computer, being a bit of a showoff, also found complex number solutions.

Next, to decide what kind of critical point this is, calculate the second derivatives.

\[
\text{fxx} := \text{diff}(fx, x);
\]

\[
fxx := \frac{16}{x^3}
\]  
(6)

\[
\text{fyy} := \text{diff}(fy, y);
\]

\[
fyy := \frac{2}{y^3}
\]  
(7)

\[
\text{fxy} := \text{diff}(fx, y);
\]

\[
fxy := 1
\]  
(8)

\[
\text{fyx} := \text{diff}(fy, x);
\]

\[
\text{fyx} := 1
\]  
(9)

\[
\text{HessianMatrix} := \text{matrix}([[\text{fxx}, \text{fxy}], [\text{fyx}, \text{fyy}]])
\]

\[
\text{HessianMatrix} := \begin{bmatrix}
\frac{16}{x^3} & 1 \\
1 & \frac{2}{y^3}
\end{bmatrix}
\]  
(10)

\[
\text{HessianDeterminant} := \text{fxx*fyy} - \text{fxy*fyx};
\]

\[
\text{HessianDeterminant} := \frac{32}{x^3 y^3} - 1
\]  
(11)

Check the sign of the Hessian determinant at the critical point...

\[
\text{subs(x=4, y=1/2, HessianDeterminant)};
\]

\[
3
\]  
(12)
We see the determinant is positive \(\Rightarrow\) definitely a local max or local min. The second derivative \(f_{xx}\) is positive, so the surface is concave up at the critical point. We conclude that there is just one critical point, \((4, 1/2)\), and at that point, the function \(f\) has a local minimum.

Let's graph the surface to see if the picture is consistent with our calculation (which it must be, but it is reassuring to see that \(1+1=2\) again).

\[
> \text{plot3d}(f, x=4-1..4+1, y=(1/2)-1..(1/2)+1, \text{axes}=\text{boxed});
\]

Why does this graph look so strange; and notice the range of outputs = \(10^{16}\) ????

In specifying the range of \(x\) and \(y\) values to consider, I allowed \(y\) to be 0. But the original function involves \(1/x\) and \(1/y\); so the graph "blows up" in this range of \(y\)'s. Let's try again, keeping the \(x\)'s and \(y\)'s symmetric around the critical point, but narrowing the focus:

\[
> \text{plot3d}(f, x=4-2..4+2, y=(1/2)-.25..(1/2)+.25, \text{axes}=\text{boxed}, \text{orientation} = [62, 108]);
\]
### ANOTHER EXAMPLE

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Problem 21

> \[ f = (2y^3 - 3y^2 - 36y + 2)/(1 + 3x^2) \]  
  \[ f' = \frac{2y^3 - 3y^2 - 36y + 2}{1 + 3x^2} \]  
  \[ \text{Eq. (13)} \]

> \[ \text{fx:=diff}(f,x); \]
  \[ \text{fx} := -\frac{6 \left(2y^3 - 3y^2 - 36y + 2\right) x}{(1 + 3x^2)^2} \]  
  \[ \text{Eq. (14)} \]

> \[ \text{fy:=diff}(f,y); \]
  \[ \text{fy} := \frac{6y^2 - 6y - 36}{1 + 3x^2} \]  
  \[ \text{Eq. (15)} \]

> \[ \text{SystemOfEquations:={fx=0, fy=0}}; \]
\[
\text{SystemOfEquations} := \left\{ \frac{-6 \left(2 y^3 - 3 y^2 - 36 y + 2\right) x}{(1+3 x^2)^2} = 0, \quad \frac{6 y^2 - 6 y - 36}{1+3 x^2} = 0 \right\}
\] (16)

First notice that the denominator factor \((1+3x^2)\) is never 0. So we can multiply through both equations by \((1+3x^2)\). And the minus sign on the left side of the first equation does not matter, since the right side is 0. Also divide out the coefficient 6 from both equations.

\[
> \text{NewSystem} := \{ (2y^3 - 3y^2 - 36y + 2)x/(1+3x^2)^2 = 0, \quad y^2 - y - 6 = 0 \};
\]

\[
\text{NewSystem} := \left\{ \frac{-6 \left(2 y^3 - 3 y^2 - 36 y + 2\right) x}{(1+3 x^2)^2} = 0, \quad y^2 - y - 6 = 0 \right\}
\] (17)

The (now) first equation says (by factoring)(which I did learn in junior high school) (but it included 9th grade)
\((y-3)(y+2)=0\), i.e. \(y=3\) or \(y=-2\).

Now we have to be careful and find the *corresponding* x values.

Substitute \(y=3\) into the second equation to get an equation in x alone. Substitute \(y=2\) into the second equation to get another equation in x alone. Solve for the value(s) of x corresponding to each value of y.

For \(y=3\)...

\[
> \text{SolveForX1} := \text{subs}(y=3, \text{NewSystem}[2]);
\]

\[
\text{SolveForX1} := 0 = 0
\] (18)

The denominator is never 0, so we can multiply both sides by \((1+3x^2)^2\) and get an equivalent equation:
\(-79 x = 0 \implies x=0\).

SO the point \((0, 3)\) is a critical point for this function \(f\).

For \(y=-2\)

\[
> \text{SolveForX1} := \text{subs}(y=-2, \text{NewSystem}[2]);
\]

\[
\text{SolveForX1} := 0 = 0
\] (19)

This ends up just like the first case: \(x=0\) is the only solution.

So \((0, -2)\) is the other critical point for function \(f\).

NEXT we use the second derivatives to try to decide what kinds of critical points these are.

\[
> \text{fxx} := \text{diff}(f, x); \text{fxx} := \text{simplify}(\text{fxx});
\]

\[
\text{fxx} := \frac{72 \left(2 y^3 - 3 y^2 - 36 y + 2\right) x^2}{(1+3 x^2)^3} - \frac{6 \left(2 y^3 - 3 y^2 - 36 y + 2\right)}{(1+3 x^2)^2} - \frac{6 \left(2 y^3 - 3 y^2 - 36 y + 2\right) (9 x^2 - 1)}{(1+3 x^2)^3}
\] (20)
\[
\begin{align*}
\text{fyy} & := \text{diff}(\text{fyy}, y); \\
\text{fyy} & := \frac{12y - 6}{1 + 3x^2} \\
\text{fx} & := \text{diff}(\text{fx}, y); \\
\text{fx} & := -\frac{6(6y^2 - 6y - 36)x}{(1 + 3x^2)^2} \\
\text{fy} & := \text{diff}(\text{fy}, x); \\
\text{fy} & := -\frac{6(6y^2 - 6y - 36)x}{(1 + 3x^2)^2} \\
\text{HessianMatrix} & := \text{matrix}([[\text{fxx}, \text{fxy}], [\text{fx}, \text{fyy}]]); \\
\text{HessianMatrix} & := \\
& \begin{bmatrix}
6(2y^3 - 3y^2 - 36y + 2)(9x^2 - 1) & 6(6y^2 - 6y - 36)x \\
(1 + 3x^2)^3 & (1 + 3x^2)^2 \\
-6(6y^2 - 6y - 36)x & 12y - 6 \\
(1 + 3x^2)^2 & 1 + 3x^2
\end{bmatrix}
\end{align*}
\]

\[
\text{HessianDeterminant} := \text{fxx} \times \text{fyy} - \text{fxy} \times \text{fy} ; \text{HessianDeterminant} := \text{simplify(\text{HessianDeterminant})}; \\
\text{HessianDeterminant} := \frac{6(2y^3 - 3y^2 - 36y + 2)(9x^2 - 1)(12y - 6)}{(1 + 3x^2)^4} - \\
\frac{36(6y^2 - 6y - 36)^2x^2}{(1 + 3x^2)^4} \\
\text{HessianDeterminant} := -\frac{36(4y^4 - 8y^3 + 225y^2x^2 - 69y^2 + 72yx^2 + 40y + 1314x^2 - 2)}{(1 + 3x^2)^4} \\
\]

These are pretty hairy. But notice that there are a lot of common factors flying around.

The second derivatives fxx, fxy, and fyy all have a denominator factor of \((1+3x^2)^2\). If we multiply each second derivative by this (which is always positive), we will not change the signs of any of the derivatives or of the Hessian determinant.

\[
\begin{align*}
\text{fxxAlt} & := \text{fxx} \times (1 + 3x^2) \\
\text{fxxAlt} & := \frac{6(2y^3 - 3y^2 - 36y + 2)(9x^2 - 1)}{(1 + 3x^2)^2} \\
\text{fxyAlt} & := \text{fxy} \times (1 + 3x^2) \\
\text{fxyAlt} & := -\frac{6(6y^2 - 6y - 36)x}{1 + 3x^2} \\
\text{fyyAlt} & := \text{fyy} \times (1 + 3x^2) \\
\text{fyyAlt} & := 12y - 6 \\
\text{HessAlt} & := \text{fxxAlt} \times \text{fyyAlt} - \text{fxyAlt}^2 ;
\end{align*}
\]
The sign of the Hessian(Alt) determinant won't be changed if we multiply through by \((1+3x^2)^2\); and we can lose a factor of 6. 

\[
\text{HessAlt} := \frac{6 \left(2y^3 - 3y^2 - 36y + 2\right) \left(9x^2 - 1\right) \left(12y - 6\right) - 36 \left(6y^2 - 6y - 36\right)^2 x^2}{\left(1+3x^2\right)^2}
\]

We are going to plug in two different y values, but both with x=0. So let's plug in x=0 first. 

\[
\text{SameSignAsHessian} := (2y^3 - 3y^2 - 36y + 2) \left(9x^2 - 1\right) \left(12y - 6\right) - 6 \left(6y^2 - 6y - 36\right)^2 x^2
\]

Now plug in the two different y values to see what is the sign of the Hessian and (if positive) the sign of fyyAlt at each. (I am using fyyAlt instead of fxxAlt because it looks simpler. Of course we also could reduce fxxAlt to a simpler expression with the same sign.)

Case y=3.

\[
\text{TestHessian} := \text{subs}(y=3, \text{SameSignAsHession\_AtCritPoint})
\]

\[
\text{TestHessian} := 2370
\]

\[
\text{TestFyy} := \text{subs}(y=3, \text{fyyAlt})
\]

\[
\text{TestFyy} := 30
\]

Conclude: Hessian determinant positive ==> the function definitely has a local max or min at (0,3). Then fyy positive ==> graph concave up ==> f has a local minimum at (0,3).

Case y=-2.

\[
\text{TestHessian} := \text{subs}(y=-2, \text{SameSignAsHession\_AtCritPoint})
\]

\[
\text{TestHessian} := 1380
\]

\[
\text{TestFyy} := \text{subs}(y=-2, \text{fyyAlt})
\]

\[
\text{TestFyy} := -30
\]

Conclude: Hessian determinant positive ==> the function definitely has a local max or min at (0,3). Then fyy negative ==> graph concave down ==> f has a local maximum at (0,-2).

Here is the graph:

\[
\text{plot3d}(f, x=-1..1, y=-3..4, axes=boxed, orientation = [33,81]);
\]

\[
f := \frac{2 y^3 - 3 y^2 - 36 y + 2}{1 + 3 x^2}
\]
Final comments:
(1) The Hessian criterion for functions of \( n \) variables, \( n>2 \), is a bit more complicated than this. The Hessian matrix (the derivative of the vector function \( \text{grad}(f) \)) is an \( (n \times n) \) matrix. One needs to look at the signs of a sequence of \( (n) \) different determinants, made from sub-matrices of the Hessian. The text explains this and gives you a "cookbook". You can use that for the two homework problems involving functions of 3 variables. For exam(s), you just need to know the "Hessian test" for functions of 2 variables.

(2) Introduction to the next topic (end of Section 4.2 and on to Section 4.3):
Suppose our task were to find the points on the BOUNDARY of the domain \( x=-1..1, y=-3..4 \) where \( f(x,y) \) is maximum or minimum. What then?

```plaintext
> C1:=plot3d(f, x=-1..-0.95, y=-3..4):
> C2:=plot3d(f, x=0.95..1, y=-3..4):
> C3:=plot3d(f, x=-1..1, y=-3..-2.95):
> C4:=plot3d(f, x=-1..1, y=3.95..4):
> display({C1, C2, C3, C4}, axes=boxed, orientation=[45, 45]);
```