Is the vector field a gradient field?

Here are two more examples of vector fields and how we can decide whether or not they are gradient fields (and find a scalar potential function for the one that is).

We have two ways to recognize that a vector field $F$ cannot be a gradient field: one way is to draw $F$ and try to see so-called (only in our class, by the way) "Escher curves". This is an intuitive, geometric, way to see if there are closed loops $C$ around which the work integrals of $F$ are nonzero. The more careful way is to check via calculus whether the work integrals of $F$ around closed loops are path-independent.

```
> with(plots):with(linalg):
```

```
Warning, the name changecoords has been redefined
```

```
Warning, the protected names norm and trace have been redefined and unprotected
```

Example 1.

```
> M:=x^2+sin(y);N:=2*x*cos(y);
M := x^2 + sin(y)
N := 2 x cos(y)
```

```
> F:=[M,N];
F := [x^2 + sin(y), 2 x cos(y)]
```

Does $F$ stand a chance of being the gradient of some scalar function $f(x,y)$ ??

Let's first draw $F$ and see if the answer is obvious.

```
> fieldplot(F, x=-2..2, y=-2..2, grid=[8,8], arrows=medium, scaling=constrained, color=black);
```
The picture is not very revealing, so we move on to more careful analysis.
Use the "curl test".

\[ Nx := \text{diff}(N, x) \]
\[ My := \text{diff}(M, y) \]

\[ Nx := 2 \cos(y) \]
\[ My := \cos(y) \]

Since \((Nx-My)\) is not identically zero (sure, it equals zero for occasional values of \(x,y\), but not for *all*), we conclude \(F\) **cannot** be a gradient field.

Example 2.
\[ M := x^2 + \sin(y) \]
\[ N := x \cos(y) + y \]

> \(F := [M, N] \);
\[ F := [x^2 + \sin(y), x \cos(y) + y] \]

\[
> \text{fieldplot}(F, x=-2..2, y=-2..2, \text{grid}=[8,8], \text{arrows}=\text{medium}, \\
> \text{scaling}=\text{constrained}, \text{color}=\text{black});
\]

Just looking at the picture, we don't see any obvious "Escher loops", but really this picture is no more helpful than the first one. So, again, we bring in Calculus...

\[
> \text{Nx} := \text{diff}(N, x); \text{My} := \text{diff}(M, y);
\]

\[ Nx := \cos(y) \]
\[ My := \cos(y) \]

The vector field \( F \) has passed the "curl test", in that \( (Nx-My)=0 \). Our theory says that so long as the domain of \( F \) has no holes (and here, the domain is the whole xy-plane), then \( F \) must in fact be a gradient field.

How do we find a scalar function \( f(x,y) \) whose gradient is \( F \)? We want the following to be true:

\[
> \text{diff}(f(x,y), x)=M;
\]

\[ \frac{\partial}{\partial x} f(x, y) = x^2 + \sin(y) \]

\[
> \text{diff}(f(x,y), y)=N;
\]
\[
\frac{\partial}{\partial y} f(x, y) = x \cos(y) + y
\]

Focus first on one of the equations:

What function \( f(x,y) \) has \( df/dx = x^2+\sin(y) \). We are just doing a Calc I anti-differentiation, viewing \( y \) as a constant (you might call this "partial integration", analogous to partial differentiation).

\[
\int M \, dx = \int M \, dx;
\]

\[
x^2 + \sin(y) \, dx = \frac{1}{3} x^3 + \sin(y) \, x
\]

The function \((1/3)x^2 + x*\sin(y)\) that Maple got omits the constant of integration, the "+c" from Calc I. Here, that is a big deal. If we add to \((1/3)x^2 + x*\sin(y)\) any function \( h(y) \) that is just a function of \( y \) alone (no \( x \)'s), we STILL have a function of \((x,y)\) whose derivative with respect to \( x \) is \( x^2+\sin(y) \).

So the general form of the solution to the first equation is

\[
> \text{Int}(M,x) = \text{int}(M,x);
\]

\[
\frac{\partial}{\partial y} f(x,y) = x \cos(y) + y
\]

NOW find what \( h(y) \) has to be by insisting that \( f(x,y) \) satisfy the second equation, i.e. \( df/dy = x*\cos(y) + y \)

\[
> \text{diff}(f(x,y),y)=x*\cos(y)+y;
\]

\[
x \cos(y) + \left( \frac{d}{dy} h(y) \right) = x \cos(y) + y
\]

Subtracting \( x*\cos(y) \) from both sides, we get

\[
> \text{diff}(h(y),y)=y;
\]

\[
\frac{d}{dy} h(y) = y
\]

\[
> h:=(1/2)*y^2;
\]

\[
h := \frac{1}{2} y^2
\]

(We could add any constant to this \( h(y) \) and still have a solution - this is our ultimate "+c" in finding \( f(x,y) \))

So

\[
> f(x,y):=(1/3)*x^3+x*\sin(y)+(1/2)*y^2;
\]

\[
f(x,y) := \frac{1}{3} x^3 + \sin(y) x + \frac{1}{2} y^2
\]

is a scalar function whose gradient is the given vector field \( F \).
\[ \nabla f(x, y) = [x^2 + \sin(y), x \cos(y) + y] \]

\[ F = [x^2 + \sin(y), x \cos(y) + y] \]

# END OF HANDOUT

# END OF HANDOUT