Suppose we have some surface $S$ in 3-space. What is the surface area of $S$?
More generally, if we have some density function $f(x,y,z)$ associated to points of $S$, what would be the mass of the sheet of "stuff" represented by $S$? How could we find the center of mass?

When we defined arclength and line integrals, we had the intuition that a smooth curve is a "piece of string" that is sitting in space; if we straighten the "string", we can see the arclength. We can imagine taking an actual piece of thin thread and tracing out some given curve. (There is a bit of physical deception in thinking that we can bend a straight piece of thread without changing the length, but it still seems intuitively acceptable.)
But with a surface, it also is intuitively clear that we cannot take a piece of paper (or plastic wrap) and bend it in the shape of a curved surface, without doing some stretching and/or folding. We can bend a flat piece of paper to look like a cylinder; but we cannot bend a piece of paper to look like a hemisphere, unless we also stretch the "paper". (Think of trying to make a "hat" out of paper that would fit closely on your head; woolen caps have a lot of elastic stretch.) Still, we can map a piece of paper onto a surface mathematically, and if we keep track of the stretching factor, we can make sense out of surface area, total mass, etc.

To start, we will work with the example of $S$ = a hemisphere.
The domain of this parametrization is a rectangle $R$ in phi-theta space. (The picture below has theta increasing from left to right, but phi decreases from bottom to top; so this is flipped from the "usual" orientation, in order to make more clear which parts of the rectangle are mapped to which parts of the hemisphere. The parts of the theta-phi rectangle where phi is close to $\pi/2$ are mapped to the parts of $S$ near the equator, while the parts of $R$ with phi near 0 are mapped to the upper "polar region" of $S$.)

All the little sub-rectangles of $R$ are distorted as they are mapped to $S$. The ones near phi=0 are more distorted than the ones near phi=$\pi/2$. To define and calculate the surface area of $S$ (or other surface integrals over $S$), we need to determine the stretching factor at each point.

The key is to recall that the area of a parallelogram whose sides are represented by vectors $v$ and $w$ is given by the size of the cross product, $|| v \times w ||$.

If, at some point, we fix phi and vary theta, we will trace out a curve (parametrized by theta) on $S$. On the other hand, if we fix theta and vary phi, we will trace out a different curve on $S$. The velocity vectors (so these are 3-dimensional vectors) of these two curves are tangent to the curves, and in fact tangent to $S$. These velocity vectors determine parallelograms that are very close approximation to the "curved parallelograms" that we see on $S$ that are the images of the little sub-rectangles of $R$. The area of such a tangent parallelogram is given by $|| \text{velocity of theta curve} \times \text{velocity of phi curve} ||$. These tangent parallelograms are close enough approximations to the "curved parallelograms", that we can their areas as the stretching factors.

For curves $[x(t), y(t), z(t)]$, our stretching factor was $|| <dx/dt, dy/dt, dz/dt> ||$.

For surfaces $[x(s,t), y(s,t), z(s,t)]$, our stretching factor is $|| <dx/ds, dy/ds, dz/ds> \times <dx/dt, dy/dt, dz/dt> ||$ (where here the derivatives are partial derivatives).
Let's look at a picture, where the "launching" point is theta=0, phi=Pi/4.

\[
\text{ThetaCurveOnSphere} := \left[ \frac{1}{2} \sqrt{2} \cos(\theta), \frac{1}{2} \sqrt{2} \sin(\theta), \frac{1}{2} \sqrt{2} \right]
\]

\[
\text{PhiCurveOnSphere} := \sin(\phi), 0, \cos(\phi)
\]

\[
\text{drawThetaCurveOnSphere} := \text{tubeplot}(\text{ThetaCurveOnSphere}, \theta=0..1, \text{radius}=0.03, \text{style}=\text{patchnogrid}, \text{color}=\text{green})
\]

\[
\text{drawPhiCurveOnSphere} := \text{tubeplot}(\text{PhiCurveOnSphere}, \phi=\pi/4..1/2+\pi/4, \text{radius}=0.03, \text{style}=\text{patchnogrid}, \text{color}=\text{red})
\]

\[
\text{SpherePlot} := \text{plot3d}([\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)], \phi=0..\pi/2, \theta=0..2\pi, \text{grid}=[10,10], \text{labels}=[X,Y,Z], \text{axes}=\text{none}, \text{orientation}=[-60,60], \text{lightmodel}=\text{light2}, \text{scaling}=\text{constrained})
\]

\[
\text{display} \{\text{drawThetaCurveOnSphere}, \text{drawPhiCurveOnSphere}, \text{SpherePlot}\}, \text{orientation}=[-30,60], \text{style}=\text{patchnogrid})
\]

\[
\text{GeneralThetaVelocity} := \text{diff}([\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)], \theta)
\]

\[
\text{ThetaVelocityAtBasePoint} := [-1/2\sqrt{2}\sin(0), 1/2\sqrt{2}\cos(0), 0]
\]
\[ \text{ThetaVelocityAtBasePoint} := \begin{bmatrix} 0, \frac{1}{2} \sqrt{2}, 0 \end{bmatrix} \]

\[ \text{GeneralPhiVelocity} := \text{diff}([\sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi)], \phi) \]

\[ \text{GeneralPhiVelocity} := [\cos(\theta) \cos(\phi), \sin(\theta) \cos(\phi), -\sin(\phi)] \]

\[ \text{PhiVelocityAtBasePoint} := [\cos(\pi/4), 0, -\sin(\pi/4)]; \]

\[ \text{PhiVelocityAtBasePoint} := \begin{bmatrix} \frac{1}{2} \sqrt{2}, 0, -\frac{1}{2} \sqrt{2} \end{bmatrix} \]

\[ \text{BasePoint} := [\sin(\pi/4) \cos(0), \sin(\pi/4) \sin(0), \cos(\pi/4)]; \]

\[ \text{BasePoint} := \begin{bmatrix} \frac{1}{2} \sqrt{2}, 0, \frac{1}{2} \sqrt{2} \end{bmatrix} \]

\[ \text{ThetaTangentLine} := \text{evalm}(\text{BasePoint} + t \times \text{ThetaVelocityAtBasePoint}); \]

\[ \text{ThetaTangentLine} := \begin{bmatrix} \frac{1}{2} \sqrt{2}, \frac{1}{2} t \sqrt{2}, \frac{1}{2} \sqrt{2} \end{bmatrix} \]

\[ \text{PhiTangentLine} := \text{evalm}(\text{BasePoint} + t \times \text{PhiVelocityAtBasePoint}); \]

\[ \text{PhiTangentLine} := \begin{bmatrix} \frac{1}{2} \sqrt{2} + \frac{1}{2} t \sqrt{2}, 0, \frac{1}{2} \sqrt{2} - \frac{1}{2} t \sqrt{2} \end{bmatrix} \]

\[ \text{drawThetaTangentLine} := \text{tubeplot}(\text{ThetaTangentLine}, t=0..1, \text{radius}=0.02, \text{style} = \text{patchnogrid}, \text{color} = \text{green}); \]

\[ \text{drawPhiTangentLine} := \text{tubeplot}(\text{PhiTangentLine}, t=0..1/2, \text{radius}=0.02, \text{style} = \text{patchnogrid}, \text{color} = \text{red}); \]

\[ \text{display} \{\text{drawThetaCurveOnSphere, drawPhiCurveOnSphere, SpherePlot, drawThetaTangentLine, drawPhiTangentLine}, \text{orientation}=[-50,70], \text{view}=[0..1, -0.5..0.5, 0.3..1]\} ; \]
The stretching factor at a general point $[\theta, \phi]$ is the size of the cross product of velocities...

> $v := \text{GeneralThetaVelocity}$; 

$v := [-\sin(\theta) \sin(\phi), \cos(\theta) \sin(\phi), 0]$ 

> $w := \text{GeneralPhiVelocity}$; 

$w := [\cos(\theta) \cos(\phi), \sin(\theta) \cos(\phi), -\sin(\phi)]$ 

> $d\text{AVector} := \text{crossprod}(v, w)$; 

$d\text{AVector} := \begin{bmatrix} -\cos(\theta) \sin(\phi)^2, -\sin(\theta) \sin(\phi) \cos(\phi), -\sin(\theta) \cos(\phi) \cos(\phi) - \cos(\theta)^2 \sin(\phi) \cos(\phi) \end{bmatrix}$ 

> $\text{StretchingFactor} := \sqrt{d\text{AVector}[1]^2 + d\text{AVector}[2]^2 + d\text{AVector}[3]^2}$; 

$\text{StretchingFactor} := \sqrt{\cos(\theta)^2 \sin(\phi)^4 + \sin(\theta)^2 \sin(\phi)^2 + \left(-\sin(\theta)^2 \sin(\phi) \cos(\phi) - \cos(\theta)^2 \sin(\phi) \cos(\phi)\right)^2}$ 

> $\text{StretchingFactor} := \text{simplify}(\text{StretchingFactor}, \text{trig})$; 

$\text{StretchingFactor} := \sqrt{1 - \cos(\phi)^2}$ 

Maple isn't confident that phi is a real number, so it doesn't simplify the above to just $\sin(\phi)$. But we can.

> $\text{StretchingFactor} := \sin(\phi)$; 

$\text{StretchingFactor} := \sin(\phi)$ 

(Recall the stretching factor for volume in spherical coordinates was $\rho^2 \sin(\phi)$. We have used spherical coordinates in this example to parametrize the sphere of radius 1 (i.e. fixing $\rho=1$), and we get the same Jacobian factor, but with $\rho=1$.)
> SurfaceArea := Int(Int(StretchingFactor, phi=0..Pi/2), theta=0..2*Pi);

\[
\text{SurfaceArea} := \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \sin(\phi) \, d\phi \, d\theta
\]

> value(SurfaceArea);

\[2\pi\]

If you go back and use some other value \( r \) for the radius of the sphere, and look at the whole sphere (i.e. \( \phi=0..\pi \)), you will have re-discovered the formula for the area of a sphere, area = \( 4\pi r^2 \).

Suppose we add the complication of some density function \( f(x,y,z) \) on the sphere. For example, let's use \( z \) as the "density". The integral of \( f(x,y,z)=z \) on the surface, divided by the area of the surface, will give the \( z \)-value of the centroid of the surface (that is, the average \( z \)-value on the surface).

In the parametrization \( \theta,\phi \rightarrow [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)] \), the \( z \) coordinate is just \( \cos(\phi) \). SO...

> zMoment := Int(Int(cos(\phi)*StretchingFactor, phi=0..Pi/2), theta=0..2*Pi);

\[
\text{zMoment} := \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos(\phi) \sin(\phi) \, d\phi \, d\theta
\]

> value(zMoment);

\[\pi\]

> value(zMoment/SurfaceArea);

\[\frac{1}{2}\]

By symmetry, the \( x \) - and \( y \) -coordinates of the centroid of surface \( S \) are both 0. So the centroid of this hemisphere is \( [0,0,1/2] \).

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end of handout
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