Line Integrals

1. The integral of a scalar function on a curve

If a wire, in the shape of some curve $C$, has a varying density, then the problem of computing the mass of the wire becomes the problem of defining the “integral” of a scalar function, here the density of the wire, whose domain is a curve in space. We/you have previously (in this course or earlier) defined integrals of scalar functions whose domains were line segments ($\int_{a}^{b} f(x) \, dx$), or 2-dimensional regions in the plane ($\int_{R} f(x, y) \, dA$), or 3-dimensional regions in space ($\int_{D} f(x, y, z) \, dV$). To define the integral of a scalar function over some domain that is just a curve in the plane or in 3-space, denoted $\int_{C} f(x, y) \, ds$ or $\int_{C} f(x, y, z) \, ds$, we follow the same list of steps as before:

1. Partition the domain set into small pieces of size $\Delta s$. If the pieces have different sizes, denote the size of the $i^{th}$ piece as $\Delta s_i$.

2. Pick a point in each little piece of curve (notation: denote the point picked in the $i^{th}$ piece by $p_i$).

3. Compute each $f(p_i)$, multiply that by $\Delta s_i$, and sum over all the pieces to get

$$\sum_{i} f(p_i) \Delta s_i .$$

4. Take the limit of these Riemann sums as the mesh of the partition gets finer, i.e. as all the $\Delta s_i \to 0$.

This limit is called the line-integral of function $f$ over the curve $C$.

Here is another situation where we are led to integrating a scalar function along a curve: Suppose our curve $C$ lies in the plane, and that there also is given some vector field $\mathbf{F}$ defined on the plane. At each point $p$ of $C$, we can compute how much $\mathbf{F}$ pushes/blows in the direction of movement along $C$. The direction of $C$ at a given point $p$ is given by the unit vector that is tangent to the curve $C$ at point $p$. We use $\mathbf{T}_p$ to denote this unit vector. The scalar amount we propose to study is the dot-product $\mathbf{F} \cdot \mathbf{T}_p$. 
• If the vector field \( \mathbf{F}(p) \) represents a force applied to an object at point \( p \), then the line integral
\[
\int_C (\mathbf{F}(p) \cdot \mathbf{T}_p) \, ds
\]
represents the work done by the force-field \( \mathbf{F} \) on an object that moves along curve \( C \).

• If the vector field \( \mathbf{F}(p) \) represents the velocity of a moving fluid at point \( p \), and \( C \) is a closed loop, then the line integral
\[
\oint_C (\mathbf{F}(p) \cdot \mathbf{T}_p) \, ds
\]
represents the amount of “circulation” of \( \mathbf{F} \) along \( C \), that is the strength of net rotation of \( \mathbf{F} \). (If the fluid is moving so the motion is mostly linear, such as in a river with little turbulence, then the circulation around closed loops will be small or zero; if there are whirlpools, then the circulation around a curve that delineates a whirlpool will be relatively large.)

2. Examples of setting up line integrals of scalar functions

• Suppose a wire is in the shape of a curve \( C \) that is the semi-circle \( x^2 + y^2 = 4, \, 0 \leq y \leq 2 \). Suppose \( \delta \) is a density function \( \delta(x, y) = y \). What is the total mass of the wire?

To compute a line integral, we need to parametrize the curve. Here we can use
\[
x(t) = 2 \cos(t), \quad y(t) = 2 \sin(t) \quad t = 0 \ldots \pi.
\]
The function \( f(p) \) is the given density function, \( \delta(x, y) = y = \sin(t) \).
The infinitesimal element of curve length, \( ds \), is \( ds = (\text{speed})(dt) \). The speed is
\[
\text{speed} = \sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{4 \sin^2 t + 4 \cos^2 t} = 2.
\]
So we have
\[
\int_C \delta(x, y) \, ds = \int_{t=0}^{\pi} (\sin(t))(2) \, dt = 4.
\]
• Suppose we have the same density function \( \delta(x, y) = y \), but now the curve is the parabola \( y = 5 - x^2 \), \( x = -1 \ldots 2 \). (Note: since \( x \) is between \(-1\) and \(2\), \( y = 5 - x^2 \) starts at 4, increases to 5, and then decreases to 1; in particular, \( y \geq 0 \), so our using density \( \delta = y \) does not get us into “negative density”, whatever that might mean.)

We can parametrize the curve as \( x = t, y = 5 - t^2 \), \( t = -1 \ldots 2 \). To compute the speed, observe that \( dx/dt = 1 \) and \( dy/dt = -2t \); so the speed is \( \sqrt{1 + 4t^2} \). Thus \( ds = (\text{speed}) \, dt = \sqrt{1 + 4t^2} \, dt \). And we have

\[
\int_C y \, ds = \int_{t=-1}^{2} (5 - t^2) \sqrt{1 + 4t^2} \, dt.
\]

(Maple actually is able to compute this integral, provided you believe in the inverse hyperbolic sine function.)

If someone were to give you the problem: “Suppose \( C \) is a wire in the shape of the parabola \( y = 5 - x^2 \), \( -1 \leq x \leq 2 \), made of some material with density \( \delta(x, y) = y \). Express the \( y \)-coordinate of the center of mass of the wire in terms of one or more definite integrals, you could answer:

\[
\frac{\int_{t=-1}^{2} (5 - t^2) \sqrt{1 + 4t^2} \, dt}{\int_{t=-1}^{2} \sqrt{1 + 4t^2} \, dt}.
\]

3. Examples of setting up work (or circulation) integrals of vector fields

• Suppose \( \mathbf{F} \) is the vector field \( \mathbf{F}(x, y) = y^2 \mathbf{i} + xy \mathbf{j} \). And suppose \( C \) is the parabola \( y = x^2 \), \( x=0 \ldots 3 \). Parametrize \( C \) as

\[
\mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j}, \quad t = 0 \ldots 3.
\]

So we have \( x(t) = t \) and \( y(t) = t^2 \).

The velocity vector at a point \( \mathbf{r}(t) \) is

\[
\mathbf{v}(t) = 1 \mathbf{i} + 2t \mathbf{j},
\]

so

\[
\text{speed} = ||\mathbf{v}|| = \sqrt{1 + 4t^2}
\]
and the unit tangent is
\[
T_p = \frac{v}{\|v\|} = \frac{1}{\sqrt{1 + 4t^2}} i + \frac{2t}{\sqrt{1 + 4t^2}} j.
\]

Thus the work integral becomes
\[
\int_C \mathbf{F}(p) \cdot T_p \, ds = \int_{t=0}^{3} (y^2 i + xy j) \cdot \left( \frac{1}{\sqrt{1 + 4t^2}} i + \frac{2t}{\sqrt{1 + 4t^2}} j \right) \sqrt{1 + 4t^2} \, dt.
\]

Notice that the “speed” terms cancel. In the expression for \( \mathbf{F} \), write \( x \) and \( y \) as whatever they are in terms of \( t \) for our parametrization of the curve \( C \), and we get
\[
\int_C \mathbf{F}(p) \cdot T_p \, ds = \int_{t=0}^{3} (t^2 i + t^2 j) \cdot (1i + 2tj) \, dt = \int_{t=0}^{3} t^4 + 2t^4 \, dt = \frac{729}{5}.
\]

- We see in the previous example that if we are given a parametrization of \( C \), the work integral for vector field \( \mathbf{F} \) along \( C \) is just the integral, over “time” \( t \), of \( \mathbf{F} \cdot \mathbf{v} \).

Here is one more example to hammer on that point: Let \( \mathbf{F} \) be the vector field \( \mathbf{F}(x, y) = (x + y)i + (xy)j \). Let \( C \) be the arc of the ellipse \( x = 3 \cos t, y = 4 \sin t, t = 0 \ldots \pi/2 \). Then the vector version of this parametrization is
\[
\mathbf{r}(t) = (3 \cos t)i + (4 \sin t)j,
\]
the velocity is
\[
\mathbf{v}(t) = (-3 \sin t)i + (4 \cos t)j,
\]
and we have
\[
\int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_{t=0}^{\pi/2} \mathbf{F} \cdot \mathbf{v} \, dt = \int_{t=0}^{\pi/2} (3 \cos t + 4 \sin t)(-3 \sin t) + ((3 \cos t)(4 \sin t))(4 \cos t) \, dt.
\]

(I’ll leave it to you and your computer, or table of integrals, to work this out.)