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P.E.T. JØRGENSEN and R.T. MOORE, *Operator commutation relations.*
D. Reidel Publishing Company (1984) xviii + 493 pp.
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Let X be a Banach space and denote by B(X) the Banach space of bounded linear operators on X. For any pair A,B in B(X) we have the well known formula

$$e^{tA}B\,e^{-tA} = e^{t\,\mathrm{ad}A}(B) \quad t \in R$$

where adA is the bounded linear operator on B(X) defined by (adA)(C) = AC-CA where C ∈ B(X) and where all the exponentials are defined by power series expansions.

This formula can be rewritten as an "operator commutation relation"

$$Be^{tA} = e^{tA}e^{-t\,\mathrm{ad}A}(B) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, t^k e^{tA}(\mathrm{ad}A)^k(B).$$

In their book the authors consider generalizations of this formula to unbounded operators A and B on Banach spaces or even arbitrary locally convex spaces. They also treat two other types of commutation relations. One is

$$B(\lambda-A)^{-1} = \sum_{k=0}^{\infty} (-1)^k(\lambda-A)^{-k-1}(\mathrm{ad}A)^k(B)$$

where $(\lambda-A)^{-1}$ is the resolvent for λ in the resolvent set. The other one is a generalization of the two formulas :

$$B\varphi(A) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\, \varphi^{(k)}(A) \quad (\mathrm{ad}A)^k(B)$$

where φ belongs to a suitable class of analytic functions, $\varphi^{(k)}$ is the k-th derivative and $\varphi(A)$, $\varphi k(A)$ the images under a functional calculus.

Such equations are encountered in the theory of differential equations e.g. if one wants to check regularity of the solution

There are two main problems in dealing with such commutation relations. One is the existence of the left hand side of the equation. Even if A and B have a common invariant dense domain, in many cases $\varphi(A)$ will not leave this domain invariant. Here one must impose some regularity conditions on the domain and the operators. The second problem is that of the convergence of the right hand side. The requirement here is that the set of commutators $(\mathrm{ad}A)^k(B)$ is contained in a finite dimensional subspace. This is a rather strong condition which however turns out to be fulfilled in many important cases.

In the book many applications are considered in the representation theory of Lie groups and in mathematical physics. But it is in fact a contribution to a field which has applications in a wide variety of subjects : astronomy, dynamical systems, geometry, stochastic filtering ....

Also related problems are considered. An important one is the exponentiation problem of a Lie algebra of unbounded operators. This is of course closely related to the well-known classical theory. The difference lies in two facts. The authors deal with a concrete Lie algebra of unbounded operators which they want to exponentiate to a strongly continuous representation of the Lie group. The other difference is that they work with C^∞-vectors in stead of the more commonly used analytic vectors. The solution to this problem is again based on the operator commutation relations.

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