
2002d:42027 42C05 46L55 47A99

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Spectral pairs in Cartesian coordinates. (English. English summary)

J. Fourier Anal. Appl. **5** (1999), *no.* 4, 285–302.

Let $\Omega = [0,1)^d$ be a set of finite measure in $\mathbf{R}^d$. A set T is a spectrum for Ω if the exponentials $\{e_t(x) = e^{2\pi i \langle t,x\rangle}\colon\ t \in T\}$ form an orthonormal basis of $L^2(\Omega)$, and (Ω, T) is called a spectral pair. A set T is a tiling set for Ω if $\Omega + T$ is a tiling of $\mathbf{R}^d$. It was conjectured by B. Fuglede [J. Functional Analysis 16 (1974), 101–121; MR 57#10500] that a set Ω has a spectrum if and only if Ω tiles $\mathbf{R}^d$ by translations; this remains open. The authors make the complementary conjecture that if (Ω, T) is a spectral pair then T is a tiling set of translations for some other set Ω'. As evidence for this, they treat the special case where $\Omega = [0,1]^d$ is the unit d-cube in $\mathbf{R}^d$, and conjecture that (Ω, T) is a spectral pair if and only if T is a tiling set for Ω, so $\Omega' = \Omega$ in this case. They prove their conjecture in dimensions $d \le 3$. This conjecture was subsequently proved in all dimensions by A. Iosevich and S. Pedersen [Internat. Math. Res. Notices 1998, no. 16, 819–828; MR 2000d:52015] and by J. C. Lagarias, J. A. Reeds and Y. Wang [Duke Math. J. 103 (2000), no. 1, 25–37; MR 2001h:11104], by different methods. This paper also gives some constructions that build spectral sets in higher dimensions from those in lower dimensions; these constructions are compatible with tilings.

A very interesting part of this paper is the Appendix, which gives an extension of the notion of spectral pair to that of a pair of Borel measures (μ, ν) on a locally compact abelian group G and its dual group Γ, respectively. In the case above, $G = \Gamma = \mathbf{R}^d$, and interesting examples are known for certain other G. The definition is asymmetric in G and Γ, but the authors establish results showing a kind of duality between the "spectral set" μ and the "spectrum" ν in a spectral pair, by showing that every "spectral set" is a "spectrum" and vice versa. This framework may represent the right level of generality for the concept of a spectral set. They also establish an "uncertainty principle" for this kind of spectral pair (Theorem 11).

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