Let $\Omega = [0,1)^d$ be a set of finite measure in $\mathbb{R}^d$. A set $T$ is a spectrum for $\Omega$ if the exponentials $\{ e_{t}(x) = e^{2\pi i \langle t, x \rangle} : t \in T \}$ form an orthonormal basis of $L^2(\Omega)$, and $(\Omega, T)$ is called a spectral pair. A set $T$ is a tiling set for $\Omega$ if $\Omega + T$ is a tiling of $\mathbb{R}^d$. It was conjectured by B. Fuglede [J. Functional Analysis 16 (1974), 101–121; MR 57#10500] that a set $\Omega$ has a spectrum if and only if $\Omega$ tiles $\mathbb{R}^d$ by translations; this remains open. The authors make the complementary conjecture that if $(\Omega, T)$ is a spectral pair then $T$ is a tiling set of translations for some other set $\Omega'$. As evidence for this, they treat the special case where $\Omega = [0,1]^d$ is the unit $d$-cube in $\mathbb{R}^d$, and conjecture that $(\Omega, T)$ is a spectral pair if and only if $T$ is a tiling set for $\Omega$, so $\Omega' = \Omega$ in this case. They prove their conjecture in dimensions $d \leq 3$. This conjecture was subsequently proved in all dimensions by A. Iosevich and S. Pedersen [Internat. Math. Res. Notices 1998, no. 16, 819–828; MR2000d:52015] and by J. C. Lagarias, J. A. Reeds and Y. Wang [Duke Math. J. 103 (2000), no. 1, 25–37; MR 2001h:11104], by different methods. This paper also gives some constructions that build spectral sets in higher dimensions from those in lower dimensions; these constructions are compatible with tilings.

A very interesting part of this paper is the Appendix, which gives an extension of the notion of spectral pair to that of a pair of Borel measures $(\mu, \nu)$ on a locally compact abelian group $G$ and its dual group $\Gamma$, respectively. In the case above, $G = \Gamma = \mathbb{R}^d$, and interesting examples are known for certain other $G$. The definition is asymmetric in $G$ and $\Gamma$, but the authors establish results showing a kind of duality between the “spectral set” $\mu$ and the “spectrum” $\nu$ in a spectral pair, by showing that every “spectral set” is a “spectrum” and vice versa. This framework may represent the right level of generality for the concept of a spectral set. They also establish an “uncertainty principle” for this kind of spectral pair (Theorem 11).

J. C. Lagarias (Florham Park, NJ)