KADISON–SINGER FROM MATHEMATICAL PHYSICS: AN INTRODUCTION

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ABSTRACT. We give an informal overview of the Kadison–Singer extension problem with emphasis on its initial connections to Dirac's formulation of quantum mechanics.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators in $\mathcal{H}$. In the language of operator algebras, the Kadison–Singer problem asks whether or not for a given MASA $\mathcal{D}$ in $\mathcal{B}(\mathcal{H})$, every pure state on $\mathcal{D}$ has a unique extension to a pure state on $\mathcal{B}(\mathcal{H})$. In other words, are these pure-state extensions unique?

It was shown recently by Pete Casazza and co-workers that this problem is closely connected to central open problems in other parts of mathematics (harmonic analysis, combinatorics (via Anderson pavings), Banach space theory, frame theory), and applications (signal processing, internet coding, coding theory, and more).

1. INTRODUCTION

This is a contribution to the webpage for an AIM 2006 Workshop on the Kadison–Singer problem. The posted text will become a permanent Introduction for the record. The current version is written by Palle Jorgensen, following lectures at the meeting by Dick Kadison. The

Introduction stresses how the mathematical context and the problem itself grew out of conceptual issues in quantum mechanics.

While the Kadison–Singer problem from the original Kadison–Singer paper [KaSi59] arose from mathematical issues at the foundation of quantum mechanics, it was found more recently to be closely connected to a number of modern areas of research in mathematics and engineering.

Indeed C^*-algebra theory was motivated in part by the desire to make precise fundamental and conceptual questions in quantum theory, e.g., the uncertainty principle, measurement, determinacy, hidden variables, to mention a few (see for example [Em84]). The 1959 Kadison–Singer problem is and remains a problem in C^*-algebras, and it has defied the best efforts of some of the most talented mathematicians of our time. The AIM workshop was motivated by recent discoveries where it was shown that the original problem is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering, including: operator theory, Banach space theory, harmonic analysis, and signal processing. While the other parts of the present website will discuss details on that, following Kadison's presentation at the workshop, this introduction will spell out some of the original motivation behind the problem at its conception.

This little Introduction to K–S is limited in scope. Here is what I tried to do, and what I stayed away from.

A number of themes are only hinted at in passing, and they could easily be expanded into a monograph. So all one can hope for is a list of pointers, and some explanations at an intuitive level, like: "what does our mathematical definition of a *state* have to do with Dirac's ideas from physics?" This was covered in Kadison's presentation at the meeting, but is now fleshed out a little. I tried hard to limit the length of the Intro, and yet still make a little dent into the murkiness of ideas from quantum mechanics. Feynman used to say: "Anyone claiming to understand quantum mechanics should be met with skepticism!" (Quoted from memory!)

I will only recall that in the mid 1920s, the period from 1925 to 27, the pioneering papers of Heisenberg, of Schrödinger, and of Dirac shaped quantum mechanics into the theoretical framework we now teach to students in physics and mathematics; see the quote from Dirac at the end. Heisenberg's paper came first (by a few weeks) and was based on the notion of transition probabilities, transition between states which later took the form of "rays" in Hilbert space, or equivalently vector states. Via corresponding matrix entries, from this emerged what became known as "matrix mechanics." Only Heisenberg

didn't realize that his matrices were infinite. Hence later additions by Max Born and John von Neumann introduced Hilbert space and operator algebras in a systematic way that is now taken for granted.

Von Neumann's axioms agree well with Heisenberg's vision, but Schrödinger formulated his equation as a partial differential equation (PDE), generalizing the classical wave equation. This was in the context of function spaces, L^2-spaces on a classical version of phase space, and it had the appearance of being "closer" to "classical" views of physics. Schrödinger's wave functions are elements in the L^2-spaces, hence "wave mechanics."

At first it was thought that the two proposed frameworks for quantum mechanics were contradictory, one was "right," but not the other! Fortunately von Neumann quickly proved that the two versions are unitarily equivalent, and since von Neumann's paper [vNeu32] and his book [vNeu68] the concept of unitary equivalence has played and continues to play a central role.

This little Introduction does not go into technical points regarding all the more recent implications, connections and applications of the K–S idea: frames, signals, etc. Others will do that; see however the Reference Supplement at the end.

I aim at offering some intuition regarding ideas and terminology that originate in quantum physics, and in von Neumann's response to Heisenberg, Schrodinger, and Dirac. Most of this can be found in courses in functional analysis and Hilbert-space theory, and operator algebras. The trouble is that if all of this were to be done properly in the Intro, it could easily become a ten-volume book set. All that is realistic is a brief little Invitation, and even that isn't easy to do well.

I hope to clarify questions asked at the meeting. If the Intro bridges some of the diverse fields represented at the meeting, that is a help. The participants include a broad and diverse spectrum of fields from math and from signal processing, but not too much math physics. Yet, a good part of the motivation derives from Dirac's vision, and I feel that it makes sense to accentuate this part of the picture.

Another reason something like this might help is that over the years such central parts of operator theory as the spectral theorem and Heisenberg's uncertainty principle have slipped out of the curriculum in departments of both math and physics.

And yet the spectral theorem and some of Heisenberg's ideas are central to the many diverse subjects touched by the K–S problem, certainly in harmonic analysis and in signal processing.

2. Math and physics

While physics students learn of quantum mechanical *states* with reference to experiments in the laboratory, in functional analysis as it is taught in mathematics departments, states are positive linear functionals ω on a fixed C^*-algebra $\mathfrak{A}$. We will assume that $\mathfrak{A}$ contains a unit, denoted I. In this context, the conditions defining ω are $\omega\colon \mathfrak{A} \to \mathbb{C}$ linear, $\omega(I) = 1$ and $\omega(a^*a) \geq 0$, $a \in \mathfrak{A}$, which is the usual positivity notion in operator algebras. As is well known, a linear functional ω on $\mathfrak{A}$ is a state if and only if $\omega(I) = 1$ and ω has norm 1.

This characterization of states (as norm-1 elements ω in the dual of $\mathfrak{A}$ with $\omega(I) = 1$) is a lovely little observation due to Richard Arens in the early 1950s. It makes things so much easier: You can now use the simplest Hahn–Banach theorem to produce all kinds of states for special purposes.

The following little picture (cited from [Jor03]) illustrates with projective geometry the simplest instance of Pauli spin matrices, and it offers a lovely visual version of the distinction between pure states and mixed states.

Recall first that the familiar two-sphere S^2 goes under the name "the Bloch sphere" in physics circles (to Pauli, a point in S^2 represents the state of an electron, or of some spin-1/2 particle, and the points in the open ball inside S^2 represent mixed states), and points in S^2 are identified with equivalence classes of unit vectors in $\mathbb{C}^2$, where equivalence of vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} = c\mathbf{v}$ with $c \in \mathbb{C}$, $|c| = 1$. With this viewpoint, a one-dimensional projection p on $\mathbb{C}^N$ is identified with the equivalence class defined from a basis vector, say $\mathbf{u}$, for the one-dimensional subspace $p\left(\mathbb{C}^N\right)$ in $\mathbb{C}^N$. A nice feature of the identifications, for $N = 2$, is that if the unit-vectors $\mathbf{u}$ are restricted to $\mathbb{R}^2$, sitting in $\mathbb{C}^2$ in the usual way, then the corresponding real submanifold in the Bloch sphere S^2 is the great circle: the points $(x, y, z) \in S^2$ given by $y = 0$. To Pauli, S^2, as it sits in $\mathbb{R}^3$, helps clarify the issue of quantum obervables and states. Pauli works with three spin-matrices for the three coordinate directions, x, y, and z. They represent observables for a spin-1/2 particle. States are positive functionals on observables, so Pauli gets a point in $\mathbb{R}^3$ as the result of applying a particular state to the three matrices. The pure states give values in S^2. Recall that pure states in quantum theory correspond to rank-one projections, or to equivalence classes of unit vectors.

In conclusion, for this little Pauli spin model, the pure states are realized as points on the 2-sphere S^2, while the mixed states are points

in the interior, i.e., in the open ball in $\mathbb{R}^3$, centered at zero, and with radius 1.

We now turn to the question: What does the positivity part of the mathematical definition of a "state" given above have to do with physics and experiments?

The answer lies in the way Heisenberg introduced probability into quantum mechanics. We elaborate on this point in eqs. (6) and (8) below in a special case.

At an intuitive level, the states from operator algebras serve to make precise an analogy to the more familiar distribution of a random variable from classical probability theory: In this "correspondence principle", random variables correspond to selfadjoint operators, that is, operators A $(= A^*)$ generalize random variables to the non-commutative, or operator algebraic, setup of quantum mechanics. This *ansatz* is consistent, since, by the spectral theorem, every selfadjoint operator is represented up to unitary equivalence as multiplication by a real-valued measurable function in a suitable L^2-space.

Hence, in quantum mechanics, observables are selfadjoint elements in the particular C^*-algebra which is selected to model the system to be studied. Now the positivity axiom: For each (mathematical) state ω we are assigning a probability to measurements of observables prepared in experiments, i.e., in associated experimental states S (instruments, prisms, magnetic fields, etc.). The information contained in S is condensed into ω. Now for the probability distributions: Given an interval J on the real line, and given a physical state S, we must calculate the probability of measuring a quantum-mechanical observable A (e.g., position, momentum, etc.) attaining values in J when it is measured in some prescribed and prepared state S, or rather ω. When A is given, the probabilities come from the spectral theorem applied to A: that is, in the form of a direct integral decomposition as recalled in (6) below.

3. States & representations

The mathematical significance of *states* lies in their relationship to *representations*. By a *representation* of a C^*-algebra $\mathfrak{A}$ we mean a homomorphism $\pi\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$, i.e., π is linear, $\pi(ab) = \pi(a)\pi(b)$, $\pi(a)^* = \pi(a^*)$, $a, b \in \mathfrak{A}$, where $\mathcal{H}$ is some complex Hilbert space, and where $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded operators on $\mathcal{H}$. Note that the space $\mathcal{H}$ depends on π. While an abstract C^*-algebra has an inherent C^*-involution, $*$, the involution on $\mathcal{B}(\mathcal{H})$ is defined as the *adjoint*, i.e., $A \to A^*$ defined by

$$(1) \qquad \langle\, Au \mid v \,\rangle = \langle\, u \mid A^*v \,\rangle, \qquad u, v \in \mathcal{H},$$

where, as is customary in physics/quantum mechanics, $\langle \cdot \mid \cdot \rangle$ denotes the inner product in $\mathcal{H}$.

For a given state ω on a C^*-algebra $\mathfrak{A}$, there is a unique cyclic representation $(\pi_\omega, \mathcal{H}_\omega, \Omega)$, where $\mathcal{H}_\omega$ is a Hilbert space, $\Omega \in \mathcal{H}_\omega$, $\|\Omega\| = 1$, and

$$(2) \qquad\qquad \omega(a) = \langle \Omega \mid \pi_\omega(a)\,\Omega \rangle, \qquad a \in \mathfrak{A}.$$

This is the so-called Gelfand–Naimark–Segal (GNS) representation. By taking orthogonal direct sums, it follows that every C^*-algebra is faithfully represented as a subalgebra of $\mathcal{B}(\mathcal{H})$ for some ("global") Hilbert space $\mathcal{H}$.

Some of the questions on the interface of math and physics relate to choices of C^*-algebras which are right for quantum mechanics. To understand this, recall that quantum-mechanical *observables* (in the mathematical language) are selfadjoint operators A (i.e., $A = A^*$) in Hilbert space. A choice of C^*-algebra $\mathfrak{A}$ implies a choice of observables

$$(3) \qquad\qquad \mathfrak{A}_{\mathrm{sa}} := \{\, A \in \mathfrak{A} \mid A = A^* \,\}.$$

We mentioned the equivalence of Heisenberg's and Schrödinger's formulations, i.e., matrix mechanics and wave mechanics, took the axiomatic form of unitary equivalence via the Stone–von-Neumann uniqueness theorem. However this does not suffice for infinite systems. There is a second version of equivalence which is tied more directly to C^*-algebras. It enters consideration in physics when passing from a finite number of degrees of freedom to an infinite number. However, this extension is not just a curiosity, and in fact is dictated by quantum statistical mechanics, and by quantum field theory. For their axiomatic formulations, we refer to the books by David Ruelle [Rue04], and by Alain Connes [Con94]. Very briefly: in the infinite cases, it turns out that different particles are governed by different statistics, e.g., bosons and fermions. In fact, the Stone–von-Neumann uniqueness theorem is false for these infinite variants; false in the sense that the natural representations are not unitarily equivalent. Specifically, a choice of statistics automatically selects an associated C^*-algebra $\mathfrak{A}$ for the problem at hand, e.g., the C^*-algebra of the canonical commutation relations (CCRs), or the canonical anticommutation relations (CARs). In the infinite case, there are issues about passing to the limit, from finite to infinite; but when the statistics and therefore the C^*-algebra are chosen, then the relevant representations will typically not be unitarily equivalent. Nonetheless, there are uniqueness theorems that take the form of C^*-isomorphisms. As it turns out, in the infinite case, these C^*-isomorphisms are not unitarily implemented.

4. Orthonormal bases (ONB)

If the chosen Hilbert space $\mathcal{H}$ is separable, we may index orthonormal bases (ONBs) in $\mathcal{H}$ by the integers $\mathbb{Z}$, i.e., $\{\, e_n \mid n \in \mathbb{Z} \,\}$. Such a choice $\{\, e_n \mid n \in \mathbb{Z} \,\}$ of ONB fixes a subalgebra $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ of operators A which are simultaneously diagonalized by $\{e_n\}$, i.e.,

$$(4) \qquad A e_n = \lambda_n e_n, \qquad n \in \mathbb{Z}, \ \lambda_n \in \mathbb{C},$$

with the sequence (λ_n) depending on A. Hence $\mathcal{D}$ is a "copy" of $\ell^\infty(\mathbb{Z})$.

Operators of the form (4) may be written as

$$(5) \qquad A = \sum_{n \in \mathbb{Z}} \lambda_n \, |e_n\rangle \langle e_n| \,,$$

where we use Dirac's notation for the rank-1 projection E_n with range $\mathbb{C}\, e_n$.

Caution to mathematicians: Eq. (5) is physics lingo, a favorite notation of Dirac ([Dir47, Dir39] quoted in [KaSi59]). Now let us facilitate the translation from physics lingo to (what has now become) math notation. The representation in Eq. (5) is how the physicist P.A.M. Dirac thought of diagonalization. Since the "bras" and the "kets" may be confusing to mathematicians, we insert explanation.

The notation used here is called Dirac's bra-ket (inner product), or ket-bra (rank-one operator) notation; and it is adopted in the physics community, and used in physics books.

Abstract considerations of Hilbert space are facilitated by Dirac's elegant bra-ket notation, which we shall adopt. It is a terminology which makes basis considerations fit especially nicely into an operator-theoretic framework: If $\mathcal{H}$ is a (complex) Hilbert space with vectors x, y, z, etc., then we denote the inner product as a Dirac bra-ket, thus $\langle\, x \mid y \,\rangle \in \mathbb{C}$. In contrast, the rank-one operator defined by the two vectors x, y will be written as a ket-bra, thus $E = |x\rangle \langle y|$. Hence E is the operator in $\mathcal{H}$ which sends z into $\langle\, y \mid z \,\rangle x$.

The general version of the spectral theorem for selfadjoint operators A in $\mathcal{H}$ takes the following form:

$$(6) \qquad A = \int_{\mathbb{R}} \lambda \, E(d\lambda) \,,$$

where $E(\,\cdot\,)$ is a projection-valued measure defined on the sigma-algebra of all Borel subsets $\mathcal{B}$ of $\mathbb{R}$. Specifically, for each $S \in \mathcal{B}$,

$$(7) \qquad E(S)^* = E(S) = E(S)^2 \,.$$

If a vector $v \in \mathcal{H}$, $\|v\| = 1$, represents a state (in fact a *pure* state on $\mathcal{B}(\mathcal{H})$) then

$$(8) \qquad \mathcal{B} \ni S \longmapsto \langle v \mid E(S)v \rangle = \|E(S)v\|^2$$

represents the probability of achieving a *measurement* of the observable A with values in S when an experiment is prepared in the state v, written $|v\rangle$ in Dirac's terminology. If, further, the system, prepared in the state corresponding to v, is designed to produce, with certainty, λ, one of the possible values that a measurement of the observable A can yield (i.e., if the probability is 1 that a measurement of A in this state will yield λ—an idealized extreme), then v is an eigenvector for A corresponding to the eigenvalue λ. If the measurement of A in the general state ω of $\mathcal{B}(\mathcal{H})$ yields λ with certainty, we say that ω is *definite* on A. The condition for ω to be definite on the observable A is that $\omega(A^2) = \omega(A)^2$ [KaSi59].

While the three formulas (6)–(8) are innocent-looking assertions from pure mathematics, they grew out of Paul Dirac's endeavors in making precise and extending the early formulations of quantum mechanics that emerged from Werner Heisenberg, Erwin Schrödinger, and Max Born; and later the math physics schools in Göttingen and in Copenhagen.

Section 6 below elaborates on these physics connections a bit more. To summarize, the "dictionary" is as follows.

(a) Observable, e.g., momentum, position, energy, spin $\rightarrow$ Selfadjoint operator, say A in Hilbert space.
(b) State (in the mathematical formulation as a positive functional, say ω) $\rightarrow$ Design and preparation of an experiment in a laboratory, magnets, mirrors, prisms, radiation, scattering, etc.
(c) Measurement (involving in its mathematical formulation the spectral theorem as given in (6)) $\rightarrow$ Application of instruments to the observable A as it is prepared in the state ω.

Caution: Note that an observable is not a number; it is a selfadjoint operator. Because of Heisenberg's uncertainty relation, even a quantum measurement is typically not really a number. Rather, it is the recording of a probability distribution of a definite observable A which is measured in a specified state. This is what Eq. (8) is saying in the language of functional analysis and operator theory.

5. PURE STATES

Let $\mathfrak{A}$ be a C^*-algebra, and denote by $\Delta(\mathfrak{A})$ the set of all states of $\mathfrak{A}$. From functional analysis we know that $\Delta(\mathfrak{A})$ is a weak*-compact

subset of $\mathfrak{A}_1^*$:= the unit ball in the dual. By Krein–Milman, we know that $\Delta(\mathfrak{A})$ is the closed convex hull of its *extreme points*. The extreme points in $\Delta(\mathfrak{A})$ are known as the *pure states* of $\mathfrak{A}$.

The Kadison–Singer question is whether or not every pure state on $\mathcal{D}$ has a unique pure-state extension to $\mathcal{B}(\mathcal{H})$. But note that by Krein–Milman, it is only the uniqueness part of the problem that is unresolved.

Experts believe that the problem/conjecture is likely to be "negative" in the sense that there are pure states on $\mathcal{D}$ with multiple and distinct pure-state extensions to $\mathcal{B}(\mathcal{H})$. Specifically, this would mean that there are pure states $\omega_1 \neq \omega_2$ on $\mathcal{B}(\mathcal{H})$ extending the same pure state on $\mathcal{D}$. In other words, starting with such a pair of pure states on $\mathcal{B}(\mathcal{H})$, that the common restriction to $\mathcal{D}$ will define the same pure state ρ on $\mathcal{D}$. Recall that purity for states on $\mathcal{D}$ is equivalent to the multiplicative rule, $\rho(AB) = \rho(A)\rho(B)$ for all $A, B \in \mathcal{D}$.

We now recall that the question of what such possible bifurcation states ρ on $\mathcal{D}$ might possibly look like concerns special points in the Stone–Čech compactification $\beta(\mathbb{Z})$ of the integers $\mathbb{Z}$, specifically points in the corona $:= \beta(\mathbb{Z}) \setminus \mathbb{Z}$. We discuss this briefly below; and there will be much more detail in a separate chapter.

Such a negative solution, if it exists, appears to hint at a "strange" element of quantum-mechanical indeterminacy.

The problem is whether or not such a bifurcation may happen from some pure states ρ on $\mathcal{D}$.

In another of the presentations included elsewhere on the site for the workshop, the Kadison-Singer problem is analyzed starting from the familiar realization of the pure states on $\mathcal{D}$ as points in the Stone–Čech compactification $\beta(\mathbb{Z})$.

6. Concluding remarks

The notion of "purity" for states has significance in both physics and mathematics. In mathematics, pure states enter into extremality considerations, in linear programming, variational analysis, and in decomposition theory. Examples: (a) Formula (8), above, shows that numbers obtained in quantum measurements attain their extreme values at pure states. (b) The vectors (e_n) in formula (5) define pure states on $\mathcal{B}(\mathcal{H})$, in fact the simplest kind of pure states, eigenvectors; i.e., each vector e_n from (5) defines the pure state $\omega_n(\,\cdot\,) := \langle e_n \mid \cdot\, e_n \rangle$ on $\mathcal{B}(\mathcal{H})$, and for the eigenvalues we have $\lambda_n = \omega_n(A)$. (c) In contrast, consideration of continuous spectrum and of Heisenberg's uncertainty relations dictates the more elaborate formulas (6)–(8) for the most general selfadjoint operators in Hilbert space.

Referring to formula (2) (Section 3 above) from mathematics, the counterpart of pure states in the GNS-correspondence between states and representations (see, e.g., [KaRi97, Arv76]) is irreducibility for the representation. Specifically: A given state ω of a C^*-algebra $\mathfrak{A}$ is pure if and only if the corresponding cyclic representation π_ω of (2) is irreducible. And it is known that irreducible representations in physics label elementary particles. In thermodynamics, pure states label pure phases. More generally, physics is concerned with composite systems and their decompositions into elementary building blocks.

Since this picture involving the GNS-correspondence (2) includes the case when the given C^*-algebra is the group C^*-algebra of a locally compact group, convexity and direct integral theory yields an abstract Plancherel formula [Seg63] for the harmonic analysis of groups; see [Seg63] and [Dix81].

7. Editorial comment by Palle Jorgensen

This is a draft of the Introduction to the Kadison–Singer IMA website `http://www.aimath.org/WWN/kadisonsinger/`. It grew out of a workshop at the AIM institute (with NSF support) in Palo Alto in September, 2006. Part of the workshop program is the creation of a permanent AIM website for the Kadison–Singer Problem, and I was assigned to write the first draft of an Introduction. Several things guided me:

(1) Motivation and history. This means that I left out mention of current trends, and that I did not include an updated Bibliography. This is left to the other writers for the project.
(2) The exposition is close to the lecture presentation Professor R.V. Kadison gave at the 2006 IMA workshop itself.
(3) It is close to the original K–S 1959 paper, and it stresses Dirac's influence.
(4) Physics, Paul Dirac, Dirac's book, and Dirac's thinking were central to the motivations. (The write-up should be understandable to physicists, for example workers in quantum computation. It is clear that Dirac's views and notation are popular in these circles!) These concerns mean that the present K–S Introduction will not talk much about a lot of other more recent applications of the K–S ideas, for example to signal processing. Others will write about that.
(5) I wanted to bridge separate communities, pure vs. applied, math vs. physics, etc. To do that, I say a few things in ways that have

become popular with physicists, but perhaps aren't especially familiar to mathematicians.

This current draft is a "live" document and is likely to undergo a few more iterations.

Acknowledgments. We thank the organizers Pete Casazza, Richard Kadison, and David Larson for comments and especially for putting together a productive and enjoyable workshop. And we are further grateful for enlightening comments from the workshop participants, especially from Gestur Ólafsson, Vern Paulsen, and Gary Weiss. We further thank Brian Treadway for typesetting and for helpful suggestions.

> I received an early copy of Heisenberg's first work a little before publication and I studied it for a while and within a week or two I saw that the noncommutation was really the dominant characteristic of Heisenberg's new theory. It was really more important than Heisenberg's idea of building up the theory in terms of quantities closely connected with experimental results. So I was led to concentrate on the idea of noncommutation and to see how the ordinary dynamics which people had been using until then should be modified to include it.
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> —P. A. M. Dirac

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References (supplement)

An incomplete list of more recent papers regarding the Kadison–Singer problem: Its implications in mathematics and its neighboring fields.

Comments:

(a) Another link between physics and the more modern signal-processing/operator-theory versions of K–S exists via what is called *complementarity* in quantum mechanics. This is also a connection to some of the other references to papers and books on C^*-algebras.

Pairs of non-commuting operators are said to be in complementarity when the partial information computed jointly from the pair is maximal compared to the information content computed from the two individually. Example: momentum and position.

As a result, extensions of pure states on one MASA are needed because the operators are non-commuting, and therefore do not have simultaneous spectral resolutions. While the concept of complementarity dates back to Niels Bohr, it has found more recent uses in harmonic analysis, quantum information/computation, and in signal processing; see, e.g., [6, 13, 17, 19], and more on the arXiv `http://arxiv.org/`.

(b) Many and diverse papers make connections to K–S via pavings, via combinatorics, matrix theory, logic, foundations, harmonic analysis, coding, information theory, and via Banach-space theory. The idea is that pavings and a lot of other parts of the big picture are or will be covered by other authors.

(c) The paper [9] and others make the connection between K–S and the Feichtinger conjecture. The Feichtinger conjecture asserts that every bounded frame can be written as a finite union of Riesz basic sequences. There are recent results on this for Weyl-Heisenberg frames; hence the connection to complementarity.

(d) Eventually there will perhaps be a big combined bibliography, but this is a continuing community project. For now, we make do with a minimal list.

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