Coupled Systems: Theory & Examples

Coupled Cell Networks

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Reference: Golubitsky and Stewart. Nonlinear dynamics of networks: the groupoid formalism. *Bull. Amer. Math. Soc.* **43** No. 3 (2006) 305–364

Thanks

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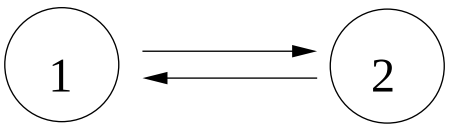
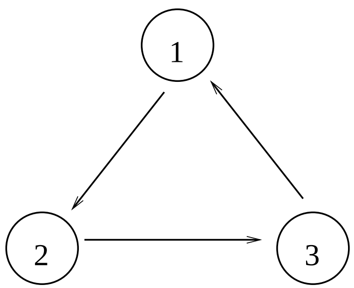
Networks and Coupled Systems



$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1) \end{aligned} \qquad x_1, x_2 \in \mathbf{R}^k$$



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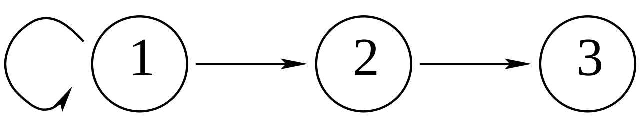

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1, \lambda) \\ \dot{x}_2 &= f(x_2, x_1, \lambda) \\ \dot{x}_3 &= f(x_3, x_2, \lambda) \end{aligned}$$

Synchrony Subspaces

- A polydiagonal is a subspace

$$\Delta = \{x : x_c = x_d \quad \text{for some subset of cells}\}$$

- A **synchrony subspace** is a flow-invariant polydiagonal

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- A **synchrony subspace** is a flow-invariant polydiagonal

- $\mathrm{Fix}(\Sigma) = \{x \in \mathbf{R}^n : \sigma x = x \quad \forall \sigma \in \Sigma\}$ is flow invariant

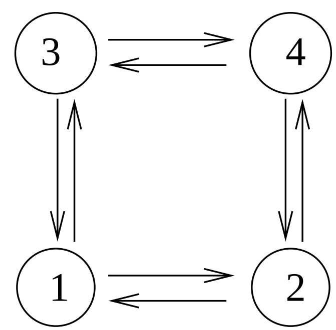
  Proof: $$f(x) = f(\sigma x) = \sigma f(x)$$

Synchrony Subspaces

- A polydiagonal is a subspace

$$\Delta = \{x : x_c = x_d \quad \text{for some subset of cells}\}$$

- A **synchrony subspace** is a flow-invariant polydiagonal
- Let $\sigma =$ be a permutation. Then $\mathrm{Fix}(\sigma)$ is a polydiagonal



- $\mathrm{Fix}((2\ 3)(1\ 4)) = \{(x_1, x_2, x_3, x_4) : x_2 = x_3; x_1 = x_4\}$
- Let Σ be a subgroup of network permutation symmetries. Then $\mathrm{Fix}(\Sigma)$ is a synchrony subspace

Coupled Cell Overview

Coupled cell system: discrete space, continuous time system

Has information that cannot be understood by phase space theory alone

symmetry	synchrony and phase shifts
network architecture	balanced colorings quotient networks

Coupled Cell Overview

Coupled cell system: discrete space, continuous time system

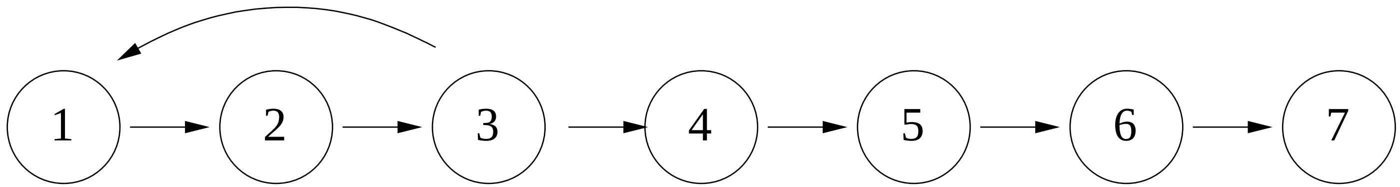
Has information that cannot be understood by phase space theory alone

- symmetry — synchrony and phase shifts
- network architecture — balanced colorings; quotient networks
- Primary Question — Which aspects of coupled cell dynamics are due to network architecture?
- Beginner Question: Are all synchrony spaces fixed-point spaces?

  Answer: No

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

Chain with Back Coupling

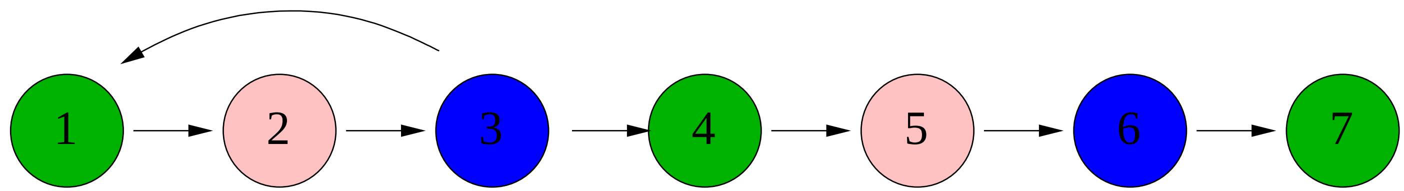


$$\dot{x}_1 = f(x_1, x_3) \qquad \dot{x}_2 = f(x_2, x_1) \qquad \dot{x}_3 = f(x_3, x_2)$$

$$\dot{x}_4 = f(x_4, x_3) \qquad \dot{x}_5 = f(x_5, x_4) \qquad \dot{x}_6 = f(x_6, x_5)$$

$$\dot{x}_7 = f(x_7, x_6)$$

Chain with Back Coupling



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$$\dot{x}_7 = f(x_7, x_6)$$

- $Y = \{x : x_1 = x_4 = x_7;\ x_2 = x_5;\ x_3 = x_6\}$ is flow-invariant
- Robust synchrony exists in networks without symmetry
- All cells are **identical within the network**; same equations

Balanced Coloring

- Let Δ be a polydiagonal
- Color equivalent cells the same color if cell coord's in Δ are equal
- Coloring is **balanced** if all cells with same color receive equal number of inputs from cells of a given color

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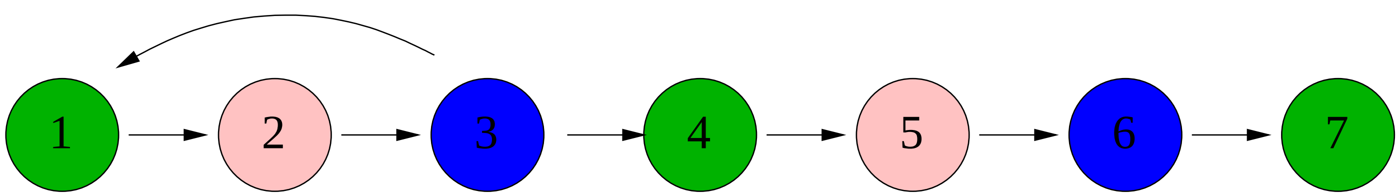

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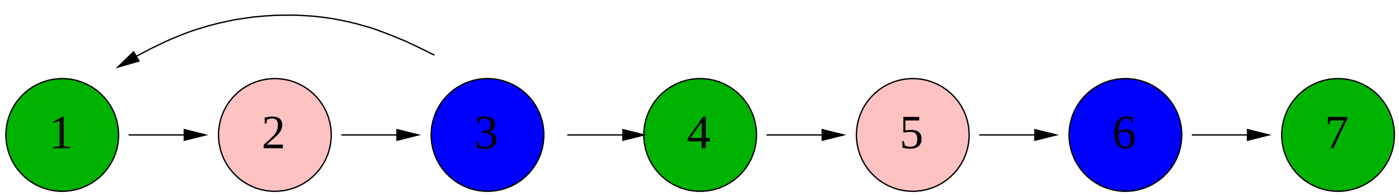
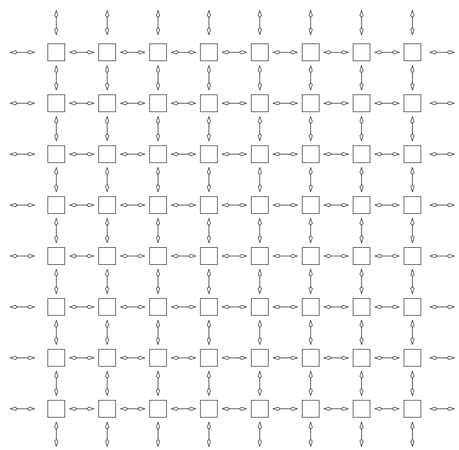


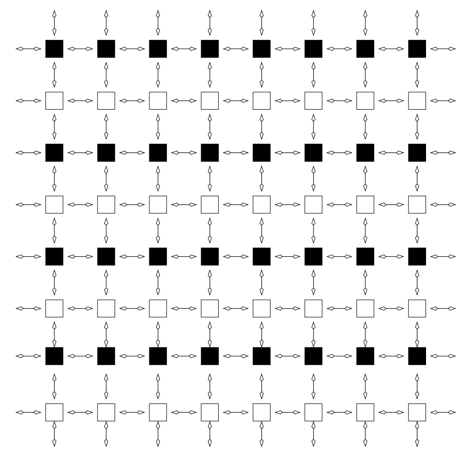
- **Theorem**: **synchrony subspace $\Longleftrightarrow$ balanced**

Stewart, G., and Pivato (2003); G., Stewart, and Török (2005)

2D-Lattice Dynamical Systems

- square lattice with nearest neighbor coupling
- Form two-color balanced relation



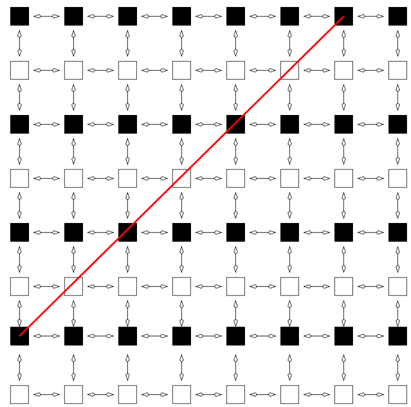
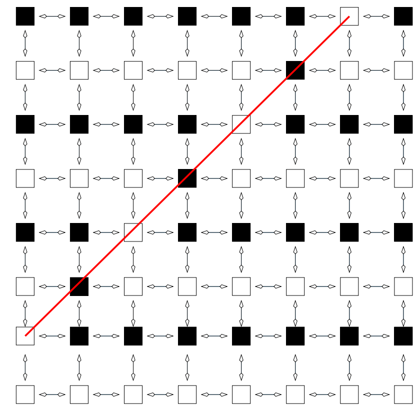


- Each black cell connected to two black and two white
  Each white cell connected to two black and two white

Stewart, G. and Nicol (2004)

Lattice Dynamical Systems

- On Black/White diagonal interchange black and white

Result is balanced

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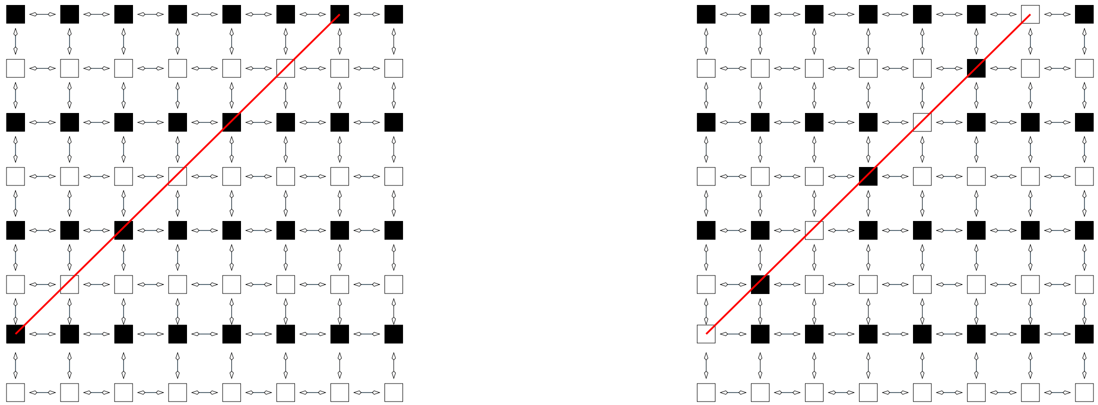
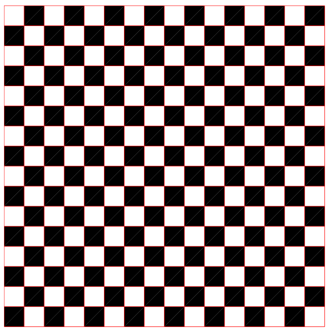


Result is balanced

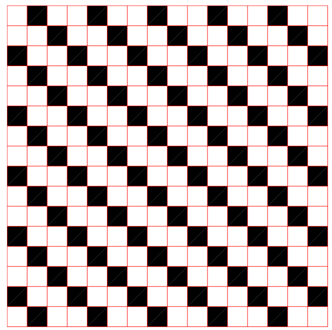
- Continuum of different synchrony subspaces

Lattice Dynamical Systems (2)

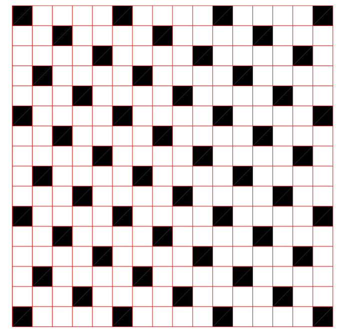
There are eight isolated balanced two-colorings on square lattice with nearest neighbor coupling



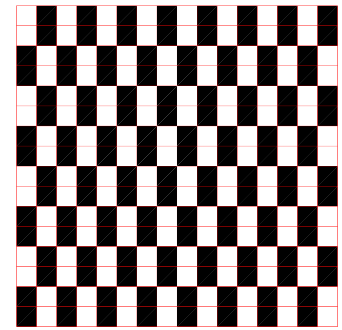
$4B \to W; 4W \to B$



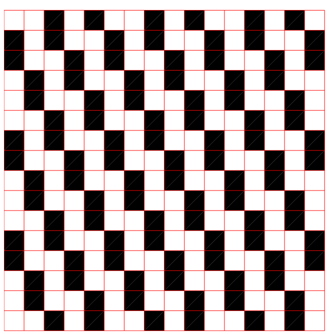
$2B \to W; 4W \to B$



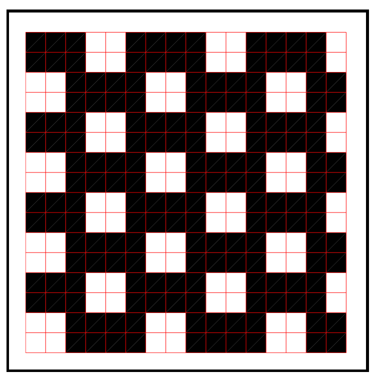
$1B \to W; 4W \to B$



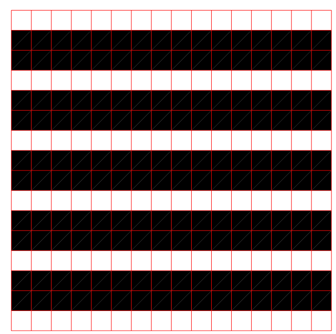
$3B \to W; 3W \to B$



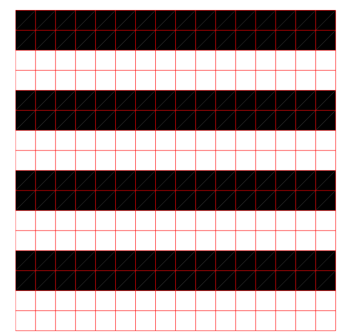
$2B \to W; 3W \to B$



$2B \to W; 1W \to B$



$2B \to W; 1W \to B$

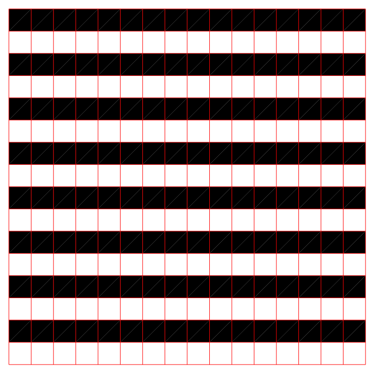


$1B \to W; 1W \to B$

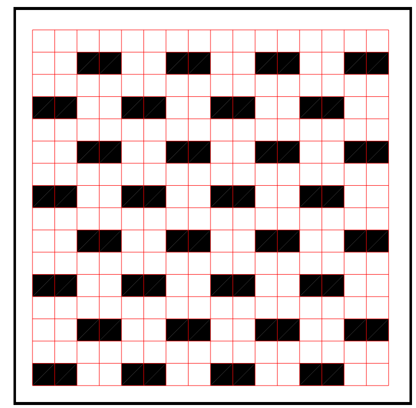
Wang and G. (2004) □ indicates nonsymmetric solution

Lattice Dynamical Systems (3)

- There are two infinite families of balanced two-colorings



$2B \to W; 2W \to B$



$1B \to W; 3W \to B$

- Up to symmetry these are all balanced two-colorings

Lattice Dynamical Systems

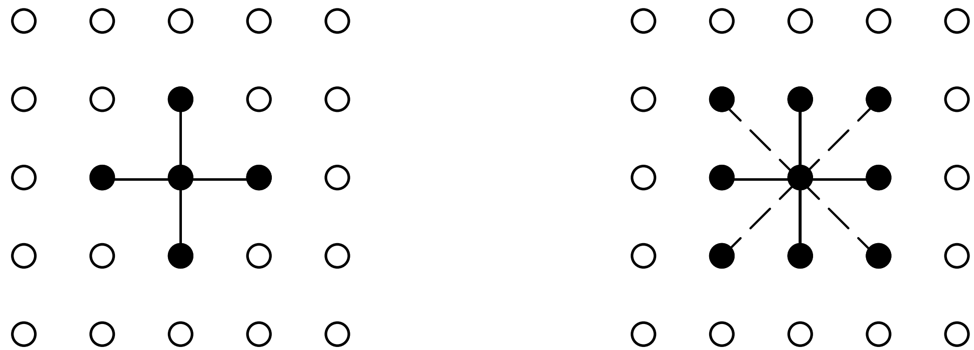
- Architecture is important

Lattice Dynamical Systems

- Architecture is important
- For square lattice with nearest and next nearest neighbor coupling
  - No infinite families
  - For each k a finite number of balanced k colorings
  - All balanced colorings are doubly-periodic

Antoneli, Dias, G., and Wang (2004)

Windows 1



$$W_0 = \{0\} \quad \text{and} \quad W_{i+1} = I(W_i)$$

- Input set of $U = I(U) = \{c \in \mathcal{C} : c$ connects to cell in $U\}$
- $\mathcal{L} = W_0 \cup W_1 \cup \cdots$
- W_{k-1} contains all k colors of a balanced k-coloring

Windows 2

- $\mathrm{bd}(U) = I(U) \setminus U$

  $c \in \mathrm{bd}(U)$ is 1-determined if color of c is determined by colors of cells in U and fact that coloring is balanced

- Define p-determined inductively

Windows 2

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- Define p-determined inductively

- All NN boundary cells are not 1-determined

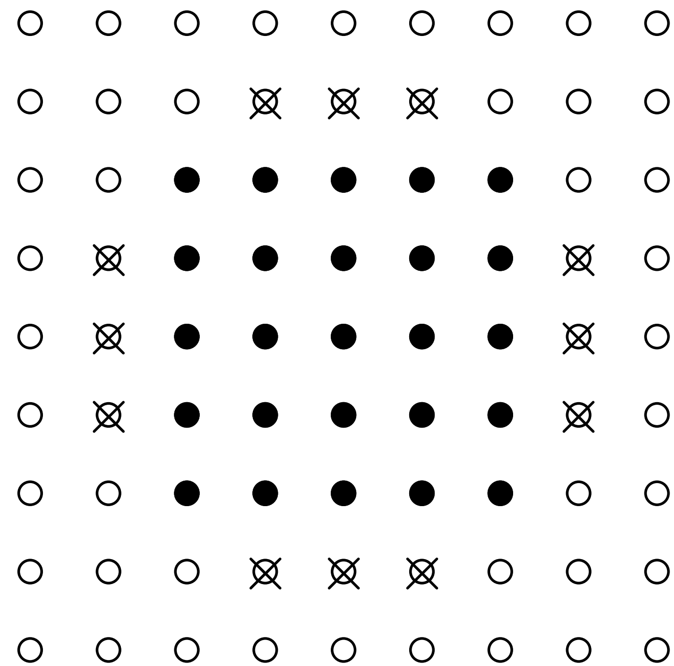
  NNN boundary cells are 1- or 2-determined

Windows 3: Square Lattice

Nearest and next nearest neighbor coupling

Black • indicates cells whose colors are known

× indicates 1-determined cells of W_2



Windows 3: Square Lattice

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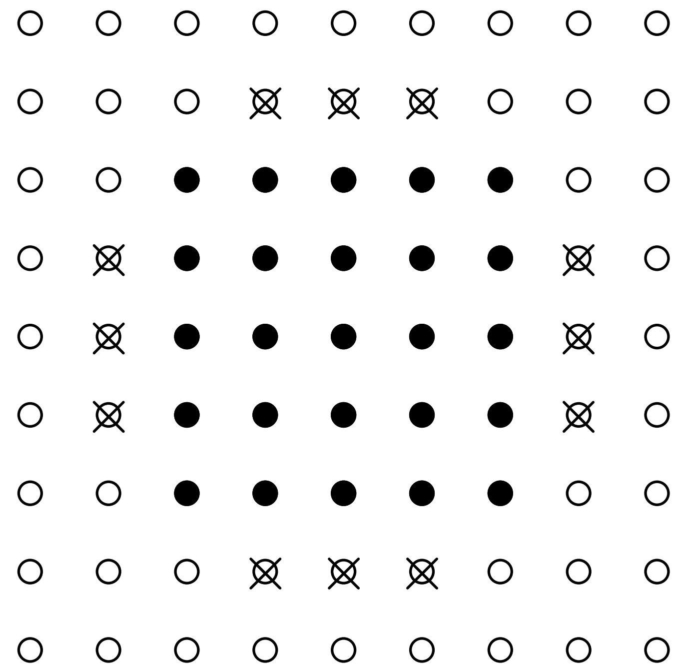

- Three cells in corners of square are 2-determined

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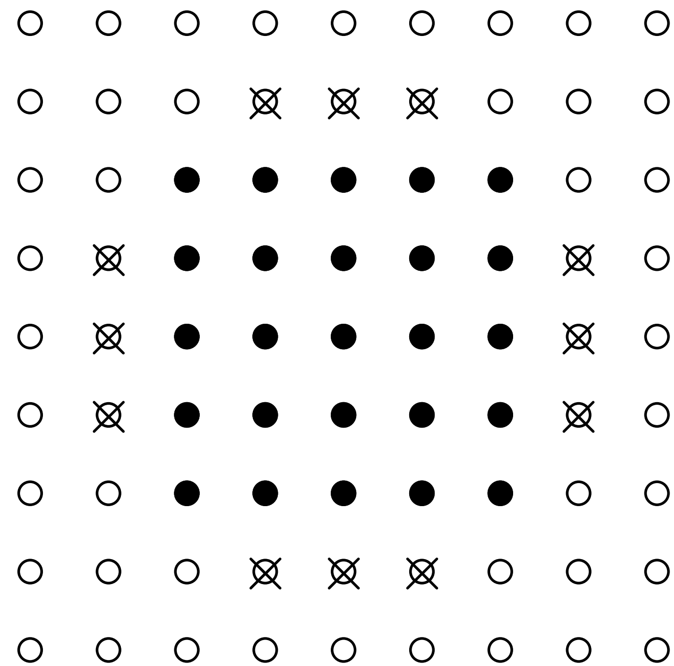
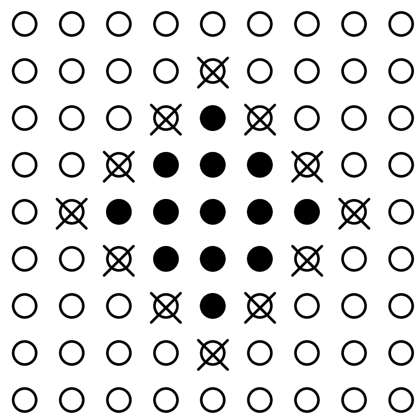


- Three cells in corners of square are 2-determined
- U determines its boundary if all $c \in \mathrm{bd}(U)$ are p-determined for some p
- W_i determines its boundary for all $i \geq 2$

Windows 4

Square lattice with Nearest neighbor coupling

W_2 is not 1-determined

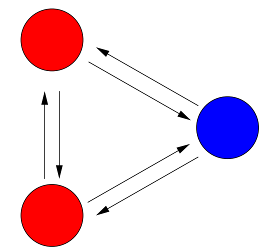


Windows 5

- W_{i_0} is a window if W_i determines its boundary $\forall\, i \geqslant i_0$
- Suppose a balanced k-coloring restricted to $\mathrm{int}(W_i)$ for some $i \geqslant i_0$ contains all k colors. Then
  - k-coloring is uniquely determined on whole lattice by its restriction to W_i
- Thm: Suppose lattice network has window. Fix k. Then
  - Finite number of balanced k-colorings
  - Each balanced k-coloring is multiply-periodic

Antoneli, Dias, G., and Wang (2004)

Quotients: Self-Coupling & Multiarrows



- Balanced two-coloring of bidirectional ring

$$
\begin{aligned}
\dot{x}_1 &= f(x_1, x_2, x_3) \\
\dot{x}_2 &= f(x_2, x_3, x_1) \\
\dot{x}_3 &= f(x_3, x_1, x_2)
\end{aligned}
$$

where $f(x, y, z) = f(x, z, y)$

Quotients: Self-Coupling & Multiarrows

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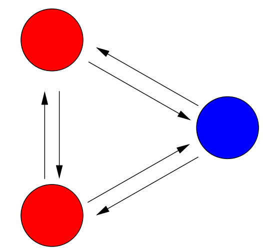

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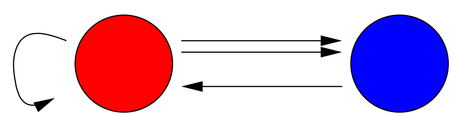
- Quotient network:



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where $f(x, y, z) = f(x, z, y)$

Quotient Networks

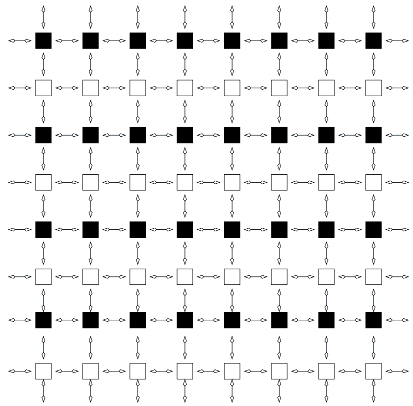
- Given cell network $\mathcal{C}$ and balanced coloring $\bowtie$
- Define *quotient network*:
  - $\mathcal{C}_{\bowtie} = \{\overline{c} : c \in \mathcal{C}\} = \mathcal{C} / \bowtie$
  - Quotient arrows are projections of $\mathcal{C}$ arrows
- Thm: Admissible DE restricts to quotient admissible DE

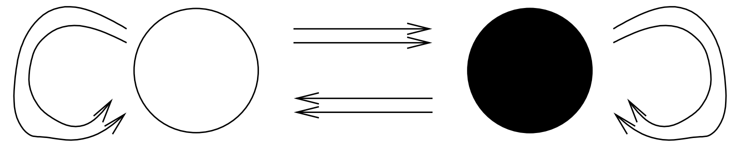
  Quotient admissible DE lifts to admissible DE

G., Stewart, and Török (2005)

Multiple Equilibria in LDE

- Recall balanced relation





- LDE on square lattice has form

$$\dot{x}_{ij} = f(x_{ij}, \overline{x_{i+1,j}, x_{i-1,j}, x_{i,j+1}, x_{i,j-1}})$$

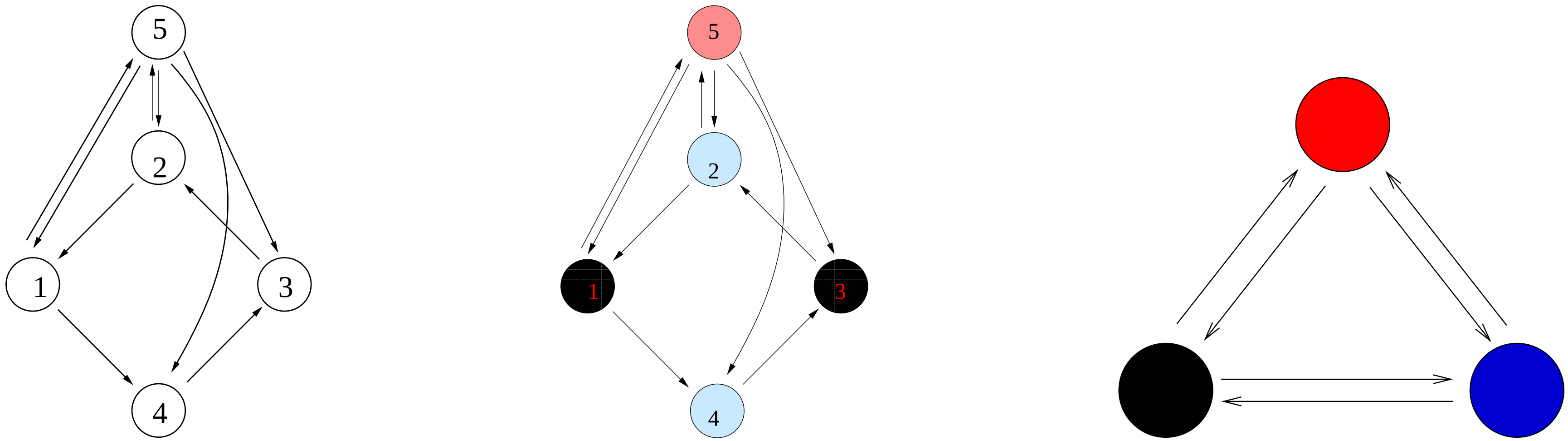
- Quotient network:

$$\begin{aligned} \dot{B} &= f(B, \overline{B, B, W, W}) \\ \dot{W} &= f(W, \overline{W, W, B, B}) \end{aligned}$$

- All quotient networks in continuum are identical

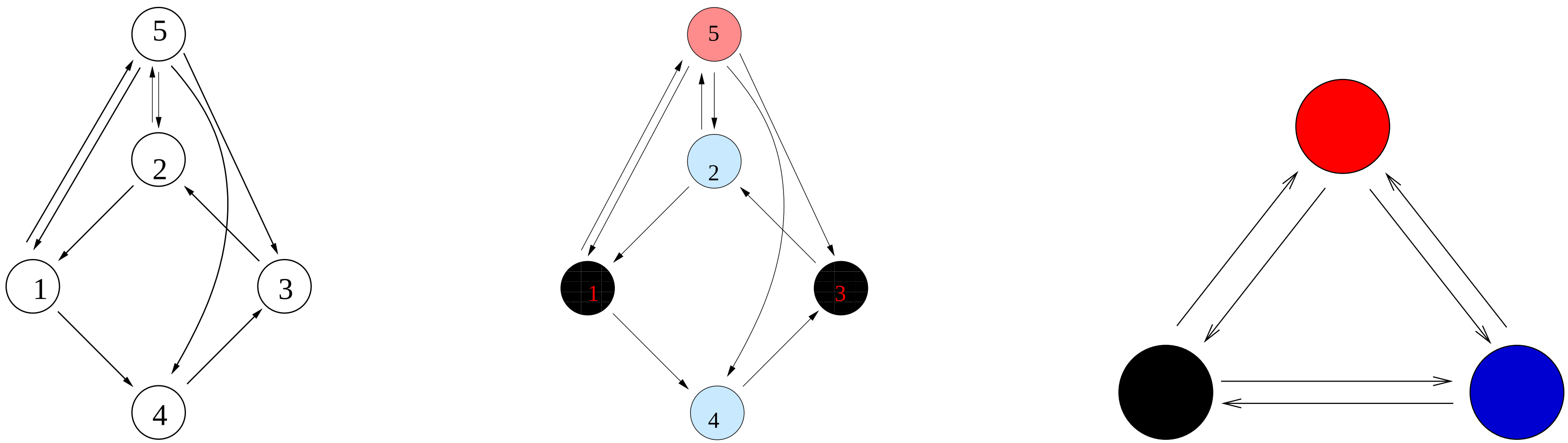
One equilibrium implies a continuum of equilibria

Asym Network; Symmetric Quotient



- Quotient is bidirectional 3-cell ring with $\mathbf{D}_3$ symmetry

Asym Network; Symmetric Quotient



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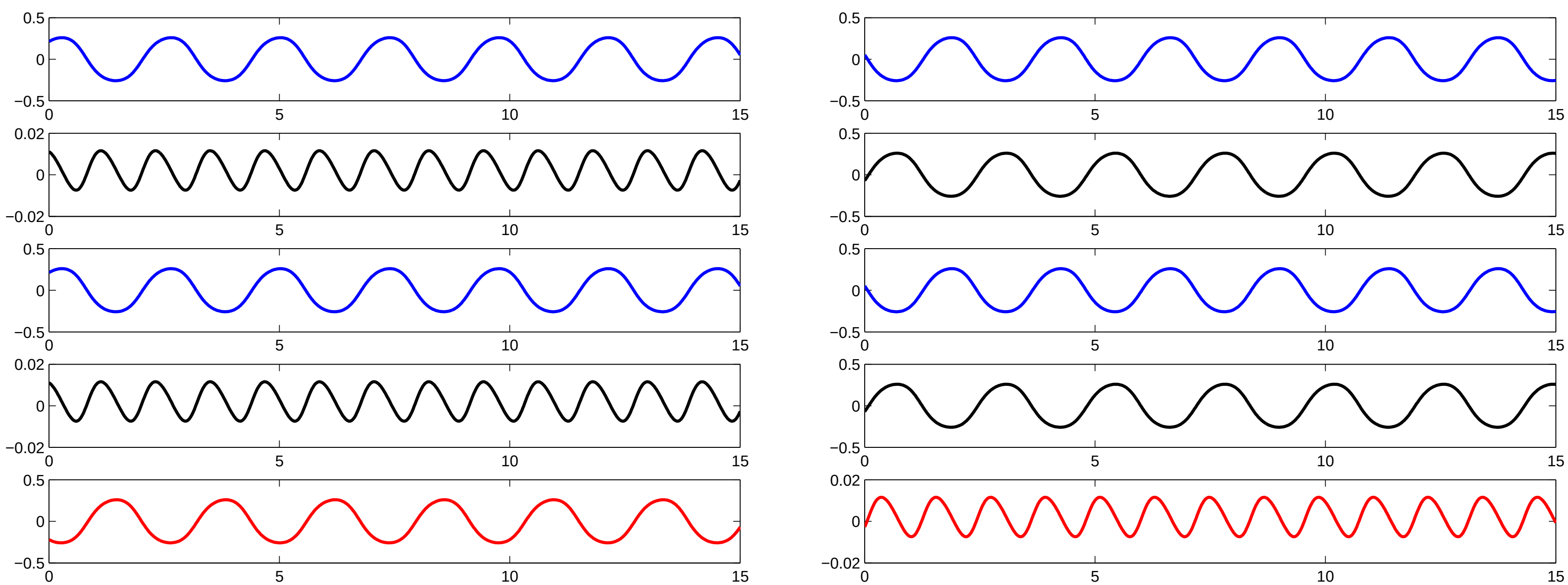
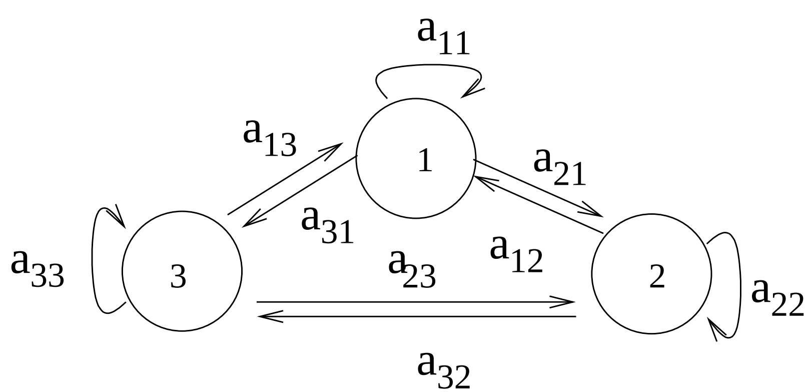


Population Models

- Cell system is homogeneous if cells are input equivalent
- Cell system has identical edges if all arrows are equivalent
- Cell system is regular if homogeneous & identical edges
- Any quotient of a regular network is regular

Regular Three Cell Networks



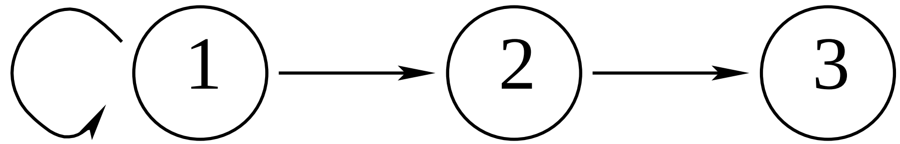
- $a_{ij} =$ number of inputs cell i receives from cell j
- *Valency* $= n =$ total number of inputs per cell

$$a_{i1} + a_{i2} + a_{i3} = n \qquad \text{for} \quad j = 1, 2, 3$$

34 regular three-cell valency 2 networks

Leite and G. (2005)

Three-Cell Feed-Forward Network



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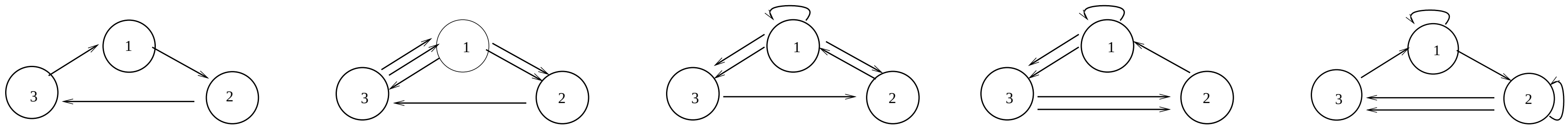
\end{aligned}
$$

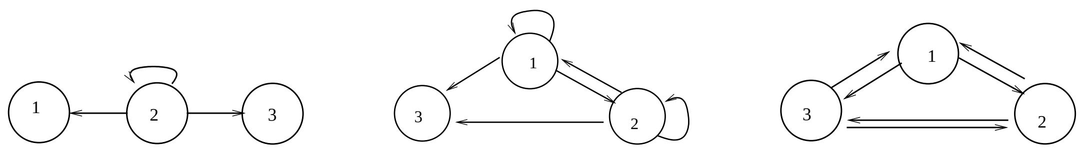
$$
J = \begin{bmatrix} \alpha + \beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}
$$

Eigenspace Types of Adjacency Matrices

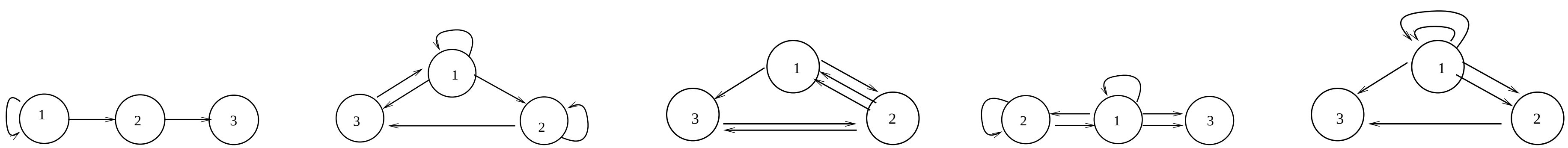
- Simple complex (no zero) eigenvalues: 2, 14, 18, 19, 24



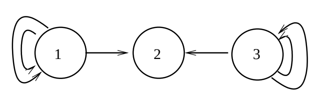
- Double with two synch-breaking eigenvectors: 4, 7, 8



- Nilpotent: 3; 6, 11, 27, 28



- Double with synchrony preserving eigenvector: 12



- Remaining 20 networks have real simple eigenvalues

Leite and G. (2006)

Jacobians and Adjacency Matrices

- Each node in regular network has ν inputs
  where $\nu =$ valency of network
- $A = (a_{ij})$ where $a_{ij} =$ number of arrows $j \to i$
  A is adjacency matrix
- ODE systems for a regular network

$$\dot{x}_j = f(x_j; \overline{x_{\sigma_j(1)}, \ldots, x_{\sigma_j(\nu)}})$$

- $x_1 = \cdots = x_n$ is flow invariant
  Can assume synchronous equilibrium
  WLOG $x_1 = \cdots = x_n = 0$ is the equilibrium
- Assume dim(internal dynamics $\equiv k = 1$)
  Jacobian $= \alpha I_n + \beta A$ where
  $\alpha =$ linearized internal node dynamics
  $\beta =$ linearized coupling

Bifurcations at Linear Level

Symmetry-breaking bifurcations

- **Theorem**:
  There is a codimension one steady-state bifurcation corresponding to each absolutely irreducible subspace

  There is a codimension one Hopf bifurcation corresponding to each irreducible subspace

Synchrony-breaking bifurcations in regular networks

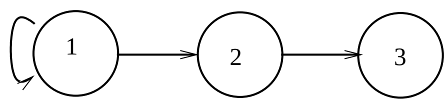
- **Theorem**: $k \geq 2$
  There is a codimension one steady-state bifurcation corresponding to each real eigenvalue of adj matrix

  There is a codimension one Hopf bifurcation corresponding to each eigenvalue of adjacency matrix

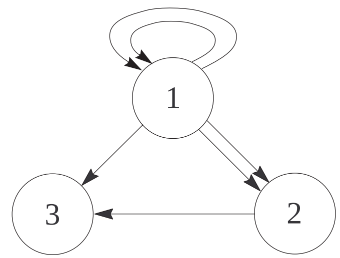
Lauterbach & G. (2009)

Nilpotent Hopf

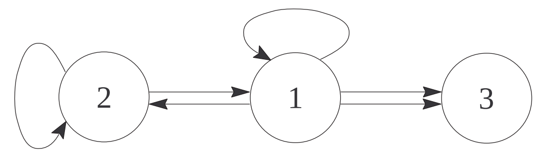
- Networks 3, 28, 27: branches that grow at $\lambda^{\frac{1}{6}}$



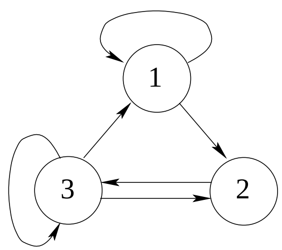
(a)

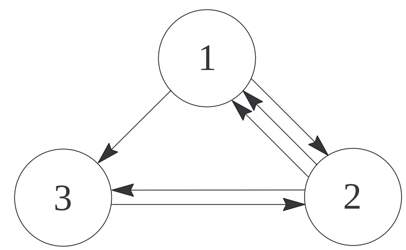


(b)

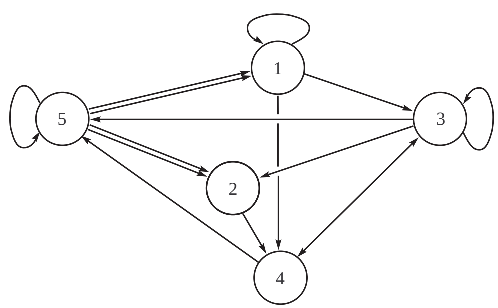


- Networks 6, 11: two or four branches that grow $\lambda^{\frac{1}{2}}$

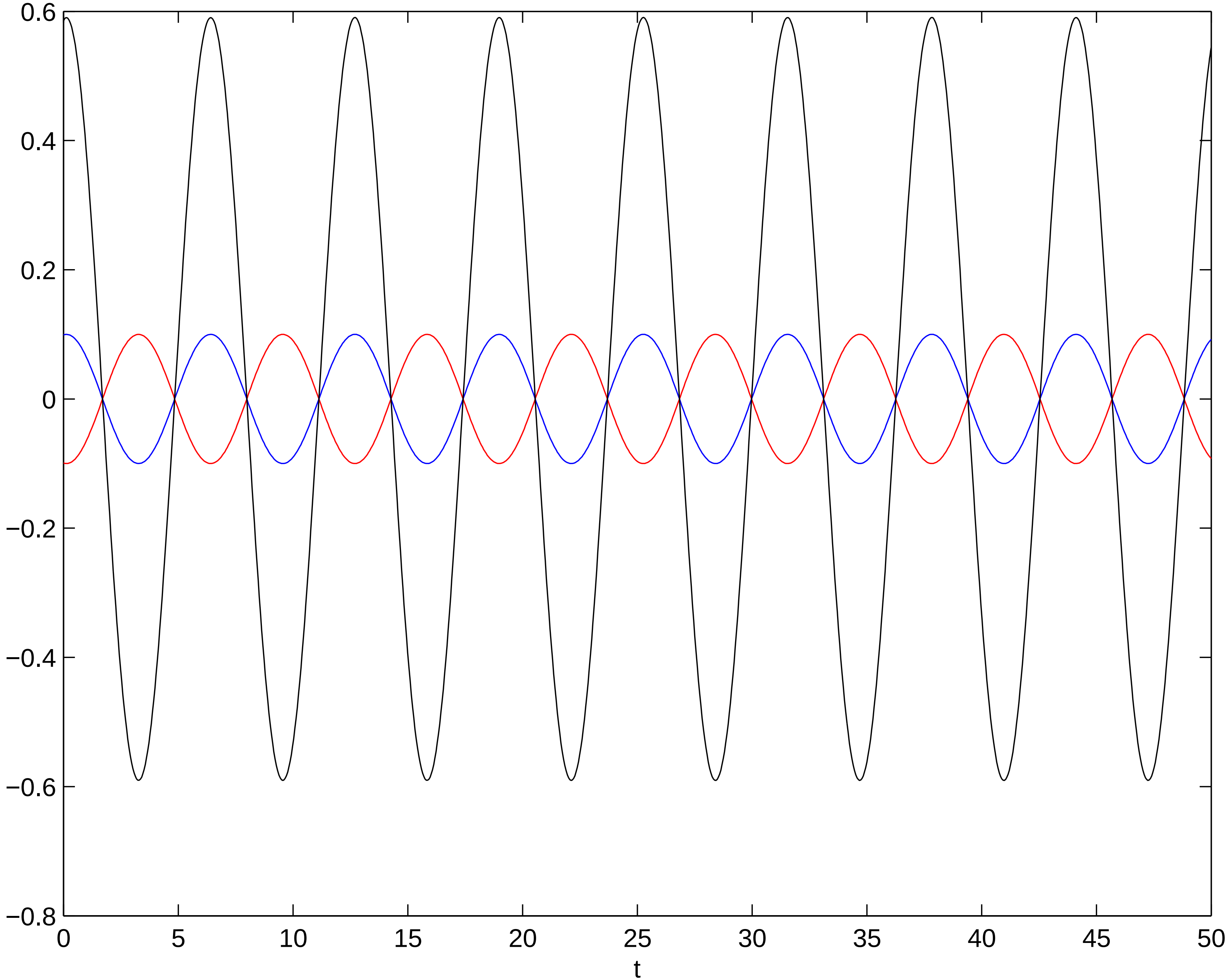




- Regular five-cell network: two branches that grow λ



Nilpotent Hopf in Network 27



Conjecture

- Number of regular networks grow superexponentially
  Number of eigenspace types grow much more slowly
- Each eigenspace type has ‘small’ number of codim 1 bifurcations — correspond to different regular networks
- **Example**: 3-4 different bifurcations for nilpotent Hopf (Elmhirst & G.)

1) Two branches: $\lambda^{\frac{1}{2}}$ and $\lambda^{\frac{1}{6}}$
2) Two branches: λ^{1}
3) Two or four branches: $\lambda^{\frac{1}{2}}$
4) Two branches: $\lambda^{\frac{1}{2}}$ and $\lambda^{\frac{1}{4}}$

Simple Zero Eigenvalue Bifurcations

- Generic equivariant case
  saddle node, transcritical, or pitchfork
- Generic network case: Not so simple
- There exist many arrow four-cell regular networks with codimension one bifurcations that are more degenerate than a pitchfork