Symmetry and Visual Hallucinations

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Klüver: We wish to stress . . . one point, namely, that under diverse conditions **the visual system** responds in terms of a limited number of form constants.

Planar Symmetry-Breaking

- Euclidean symmetry: translations, rotations, reflections
- Symmetry-breaking from translation invariant state in planar systems with Euclidean symmetry leads to
  - Stripes:
    States invariant under translation in one direction
  - Spots:

    States centered at lattice points

Sand Dunes in Namibian Desert

Mud Plains

Outline

1. Geometric Visual Hallucinations

2. Structure of Visual Cortex

   Hubel and Wiesel hypercolumns; local and lateral connections; isotropy versus anisotropy

3. Pattern Formation in V1

   Symmetry; Three models

4. Interpretation of Patterns in Retinal Coordinates

Visual Hallucinations

- Drug **uniformly** forces activation of cortical cells
- Leads to spontaneous pattern formation on cortex
- Map from V1 to retina;
  translates pattern on cortex to visual image

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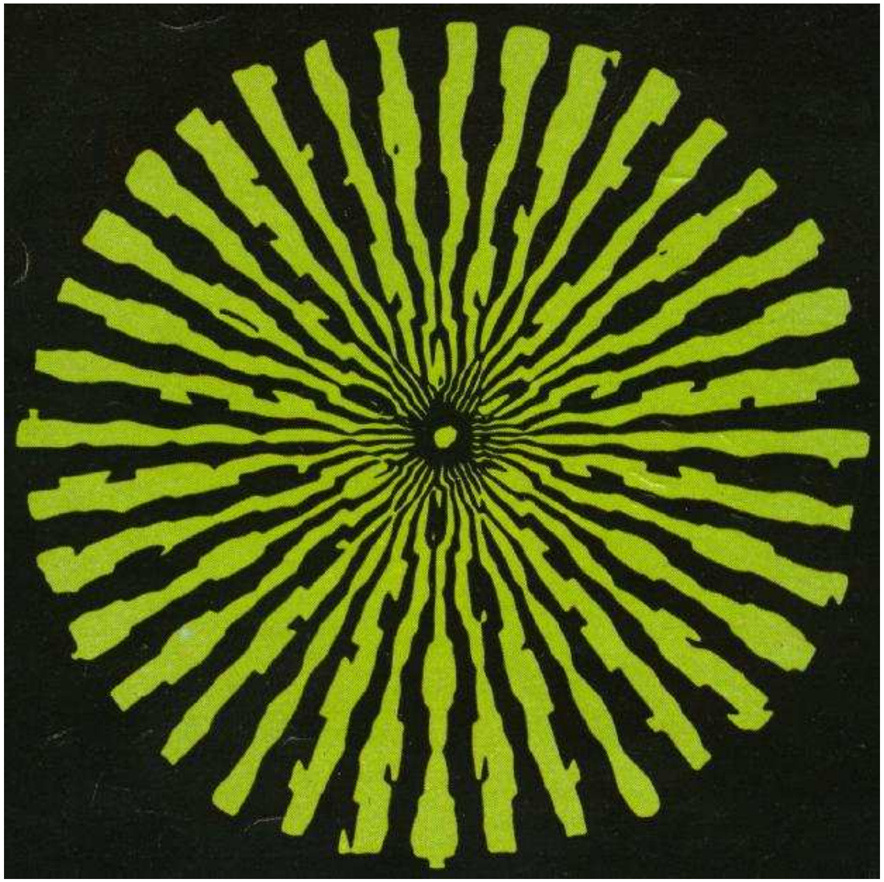
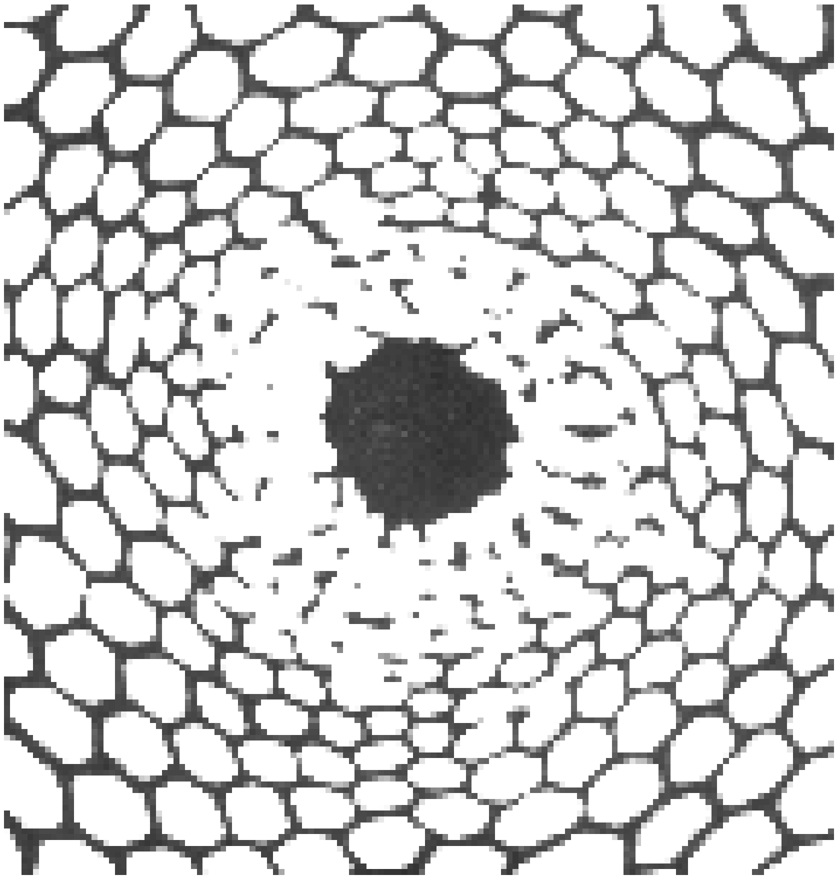
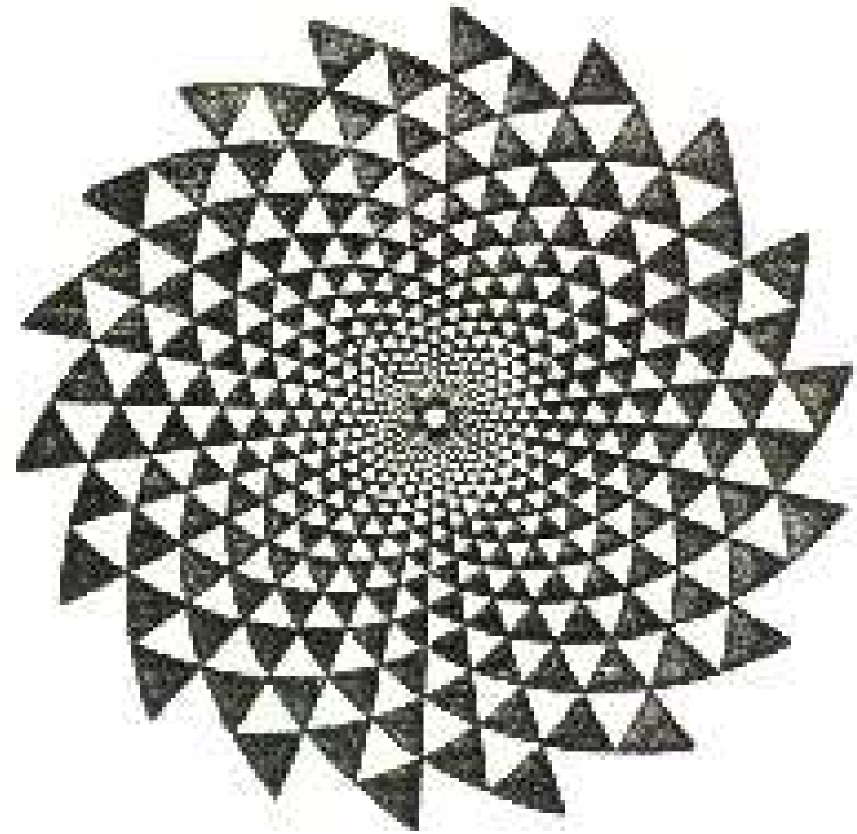
  translates pattern on cortex to visual image
- Patterns fall into four *form constants* (Klüver, 1928)
  - tunnels and funnels
  - spirals
  - lattices includes honeycombs and phosphenes
  - cobwebs

Funnels and Spirals

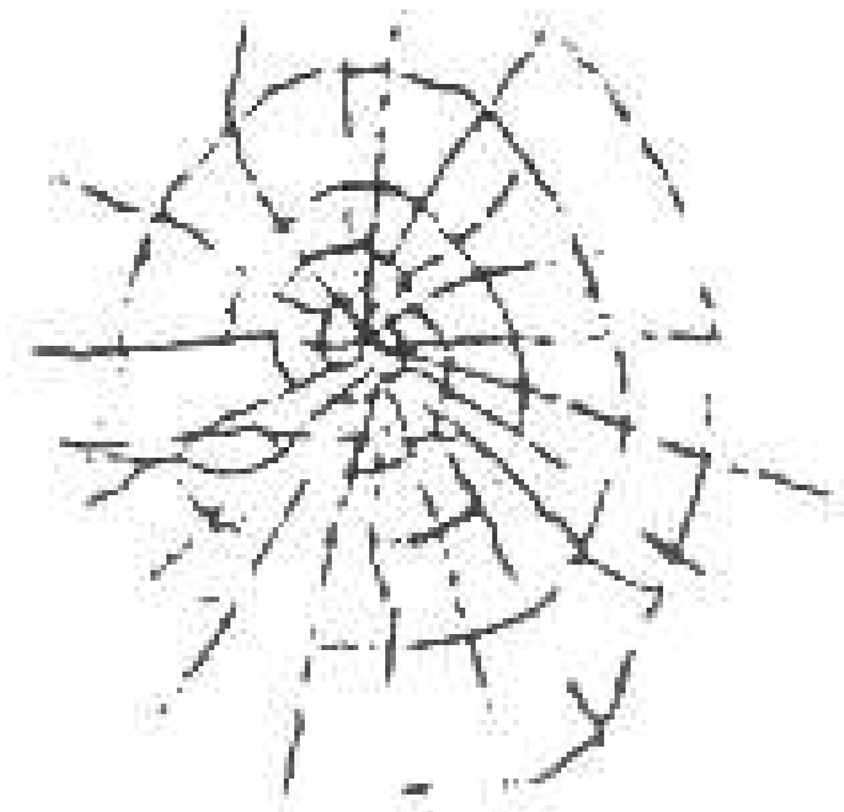


Lattices: Honeycombs & Phosphenes



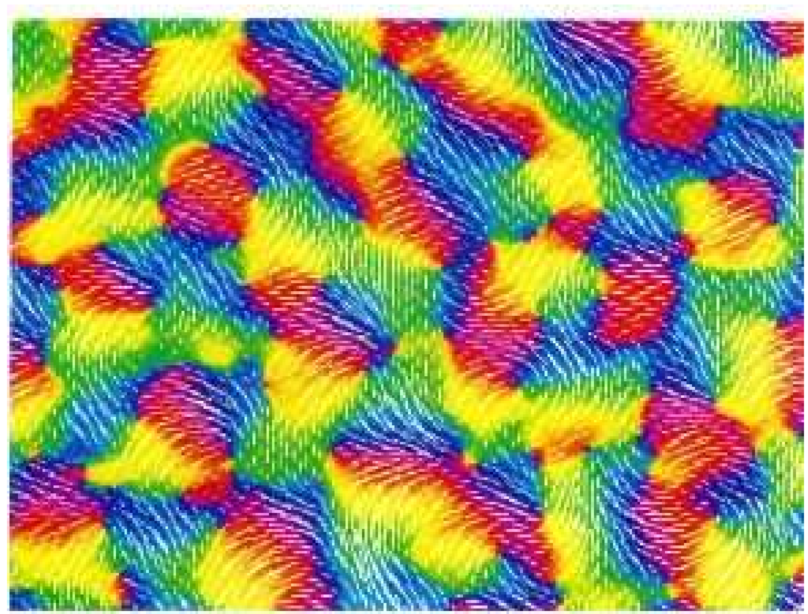


Cobwebs



Orientation Sensitivity of Cells in V1

- Most V1 cells sensitive to *orientation* of contrast edge

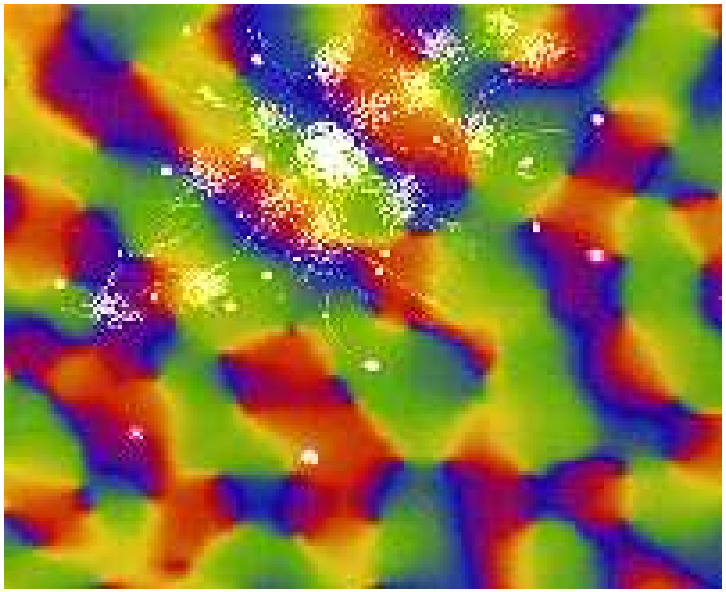


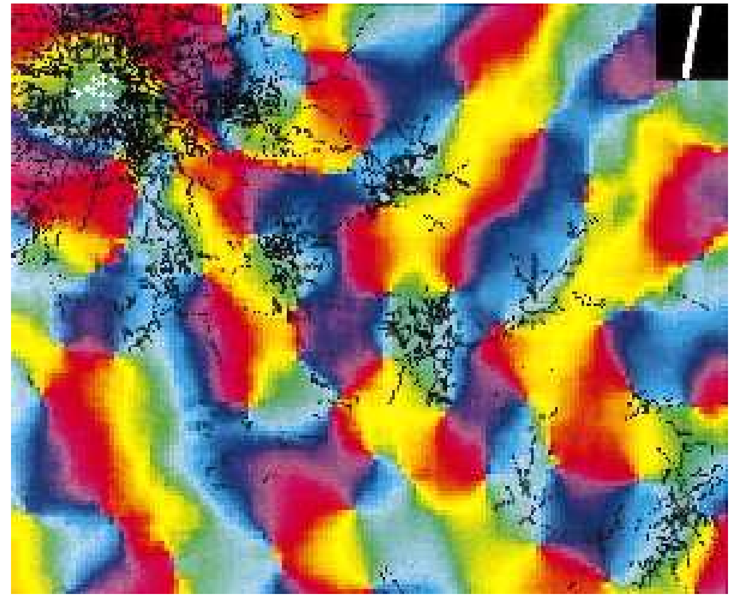
Distribution of orientation preferences in Macaque V1 (Blasdel)

- Hubel and Wiesel, 1974
  Each millimeter there is a *hypercolumn* consisting of orientation sensitive cells in every direction preference

Structure of Primary Visual Cortex (V1)

- Optical imaging exhibits pattern of connection





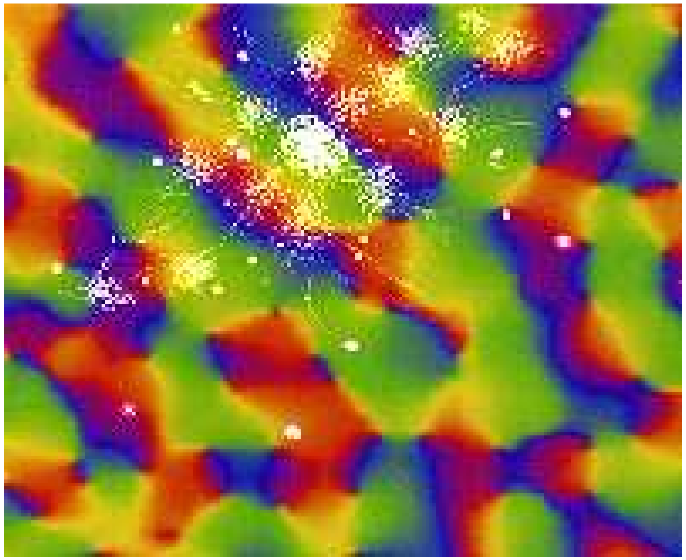
V1 lateral connections: Macaque (left, Blasdel) and Tree Shrew (right, Fitzpatrick)

- Two kinds of coupling: local and lateral

  (a) local: cells $< 1mm$ connect with most neighbors

  (b) lateral: cells make contact each mm along axons; connections in direction of cell's preference

Anisotropy in Lateral Coupling



- Macaque: most anisotropy due to stretching in direction orthogonal to ocular dominance columns. Anisotropy is weak.

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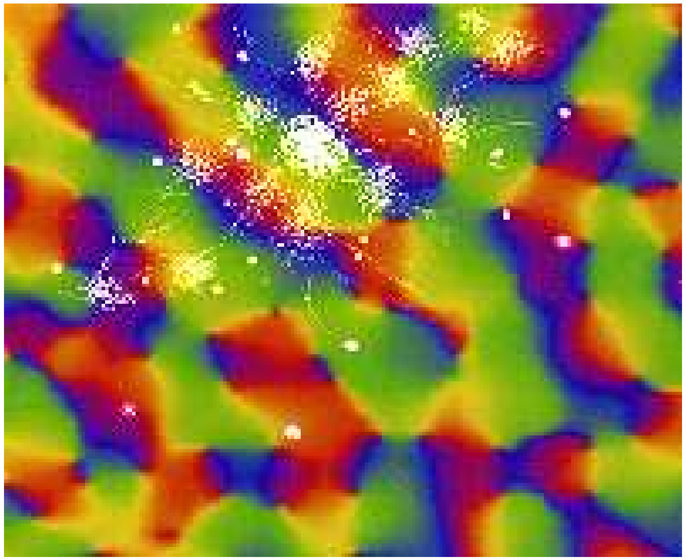

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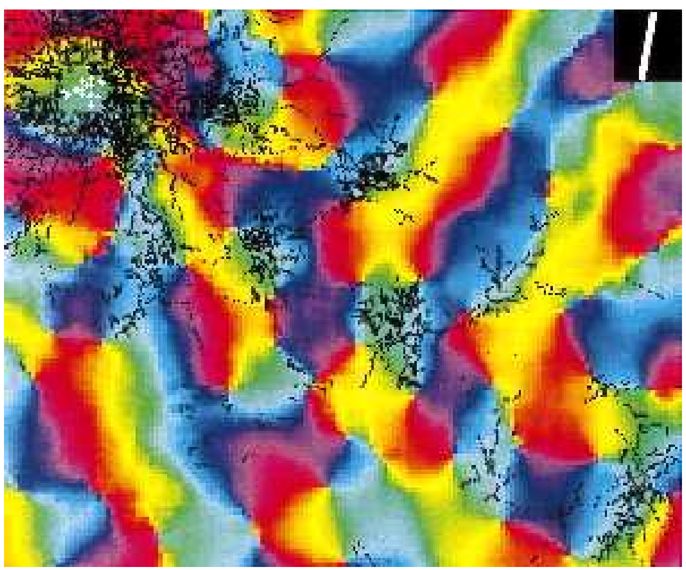
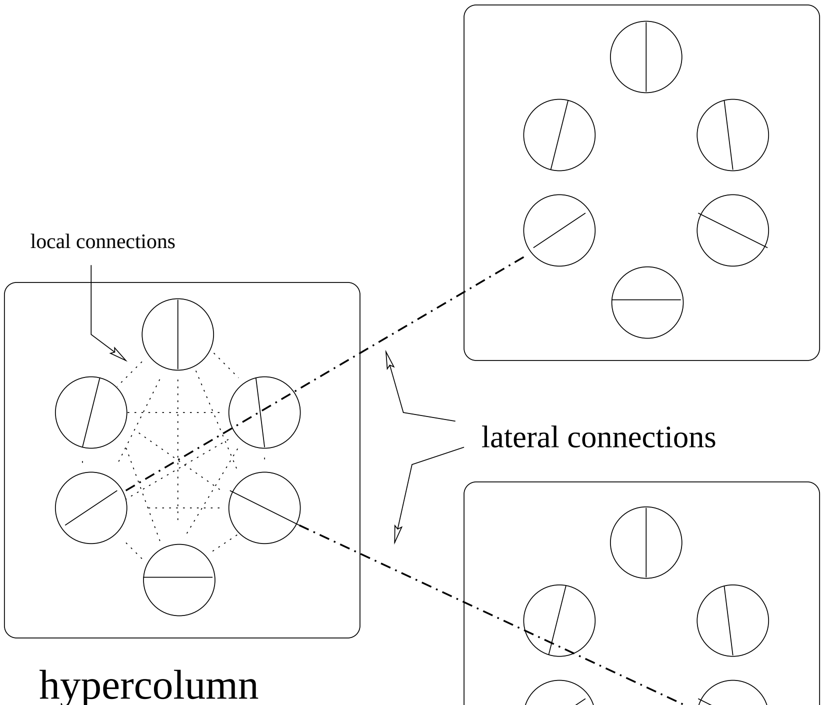


- Tree shrew: anisotropy pronounced

Action of Euclidean Group: Anisotropy



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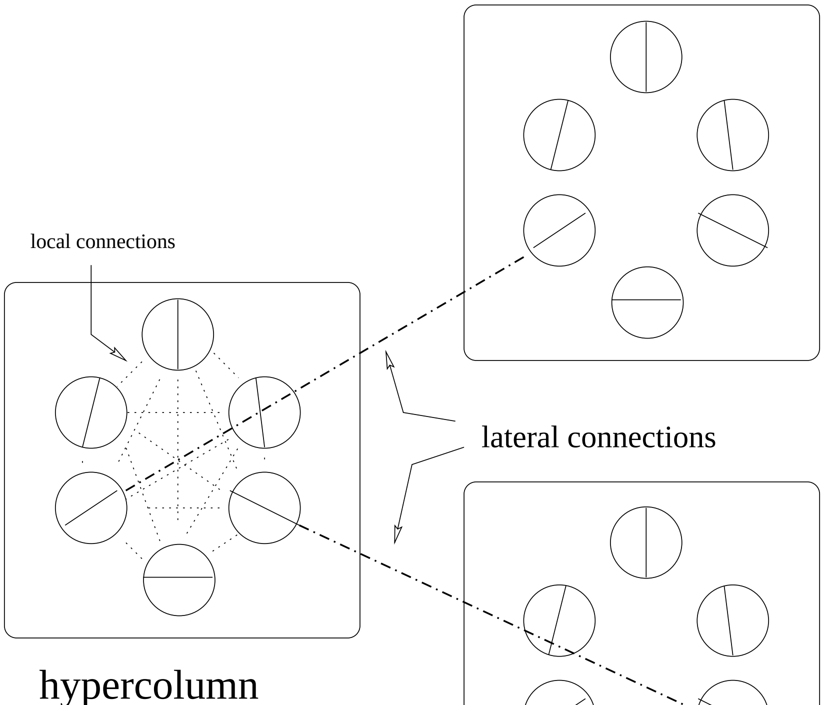
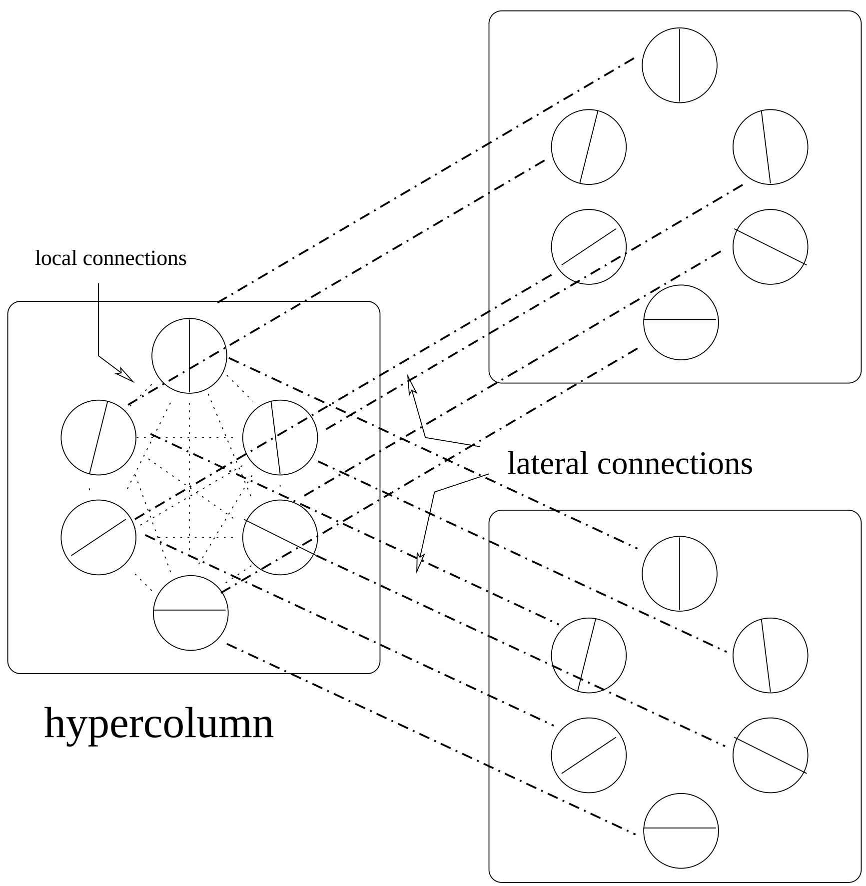


- Abstract physical space of V1 is $\mathbf{R}^2 \times \mathbb{S}^1$ — not $\mathbf{R}^2$
  Hypercolumn becomes circle of orientations
- Euclidean group on $\mathbf{R}^2$: translations, rotations, reflections
- Euclidean groups acts on $\mathbf{R}^2 \times \mathbb{S}^1$ by

$$\begin{aligned} T_y(x,\varphi) &= (T_y x, \varphi) \\ R_\theta(x,\varphi) &= (R_\theta x, \varphi + \theta) \\ \kappa(x,\varphi) &= (\kappa x, -\varphi) \end{aligned}$$

Isotropic Lateral Connections



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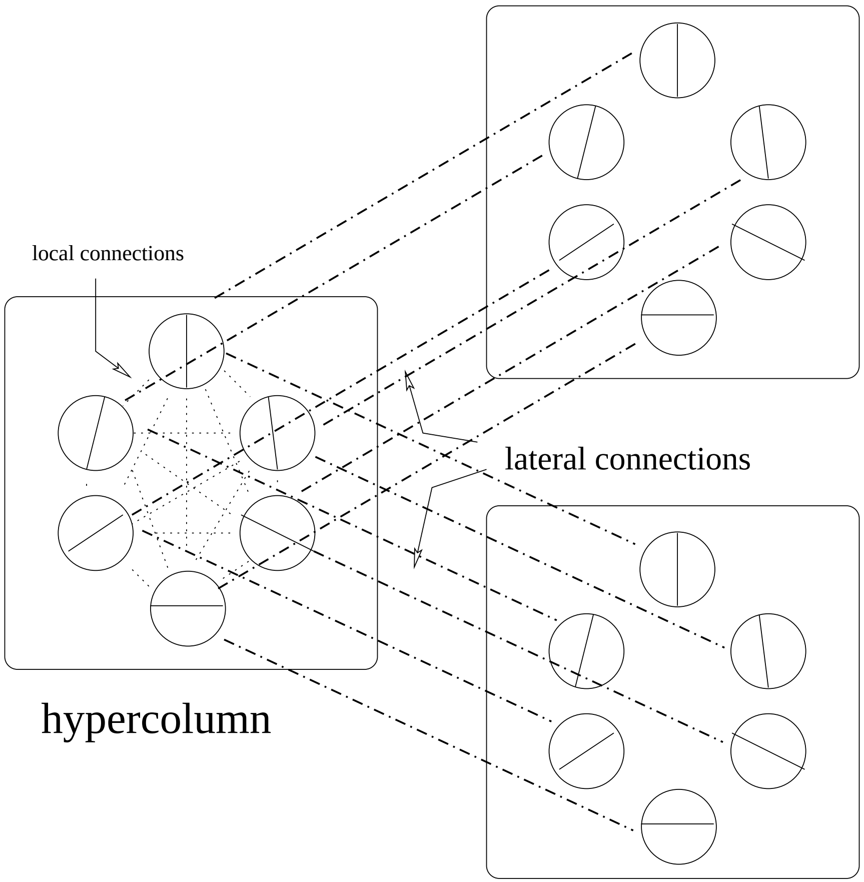

- New $\mathbf{O}(2)$ symmetry

$$\hat{\phi}(x, \varphi) = (x, \varphi + \hat{\phi})$$

- Weak anisotropy is forced symmetry breaking of

$$\mathbf{E}(2)\dot{+}\mathbf{O}(2) \rightarrow \mathbf{E}(2)$$

Three Models

- $\mathbf{E}(2)$ acting on $\mathbf{R}^2$ (Ermentrout-Cowan)
  neurons located at each point x
  Activity variable: $a(x) =$ voltage potential of neuron
  Pattern given by threshold $a(x) > v_0$

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- Shift-twist action of $\mathbf{E}(2)$ on $\mathbf{R}^2 \times \mathbf{S}^1$ (Bressloff-Cowan)
  hypercolumns located at x; neurons tuned to φ
  strongly anisotropic lateral connections
  Activity variable: $a(x, \varphi)$
  Pattern given by winner-take-all

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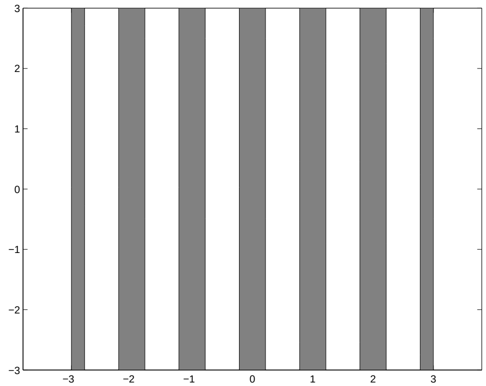
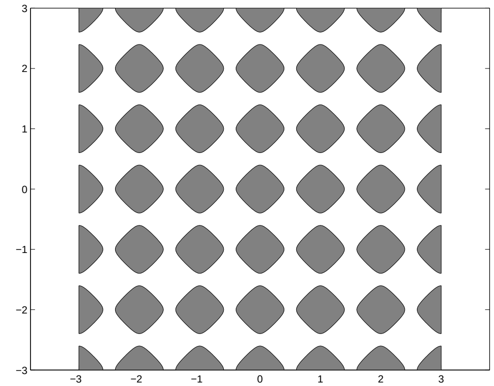
- Symmetry breaking: $\mathbf{E}(2)\dot{+}\mathbf{O}(2) \to \mathbf{E}(2)$
  weakly anisotropic lateral coupling
  Activity variable: $a(x, \varphi)$
  Pattern given by winner-take-all

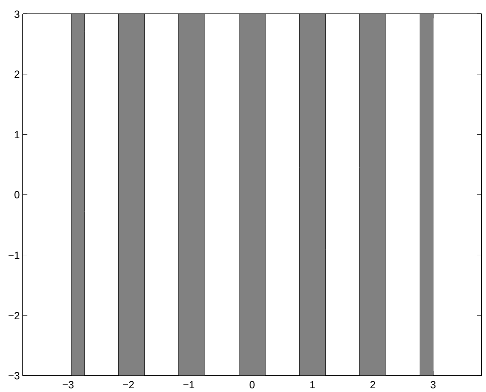
Planforms For Ermentrout-Cowan

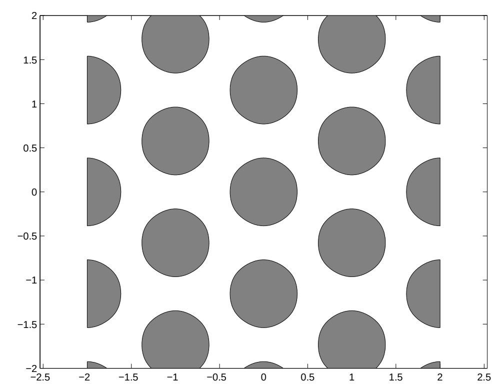
Threshold Patterns





Square lattice: stripes and squares





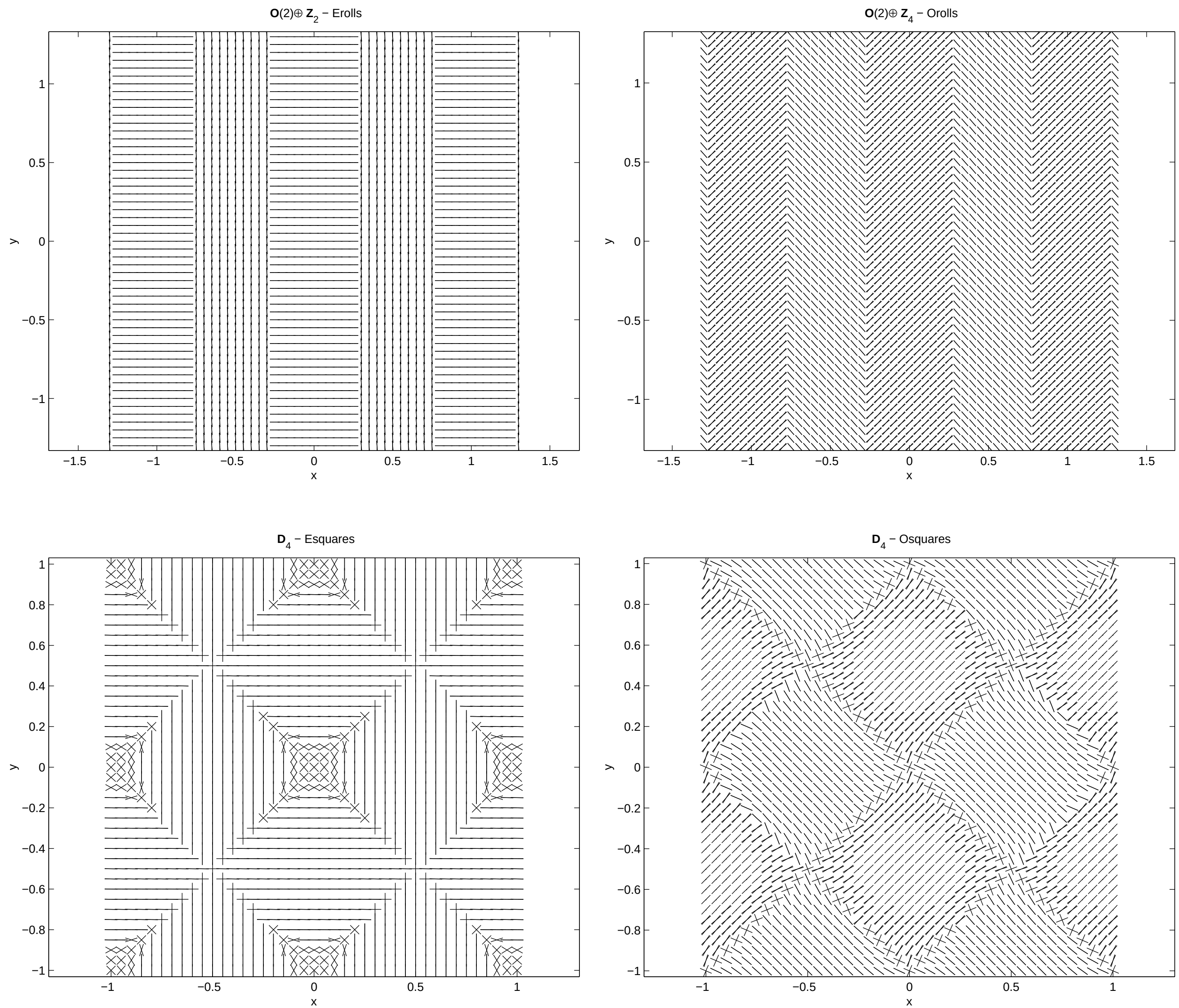
Hexagonal lattice: stripes and hexagons

Winner-Take-All Strategy

Creation of Line Fields

- Given: Activity $a(\mathbf{x}, \varphi)$ of neuron in hypercolumn at $\mathbf{x}$ sensitive to direction φ
- Assumption: Most active neuron in hypercolumn suppresses other neurons in hypercolumn
- Consequence: For all $\mathbf{x}$ find direction $\varphi_{\mathbf{x}}$ where activity is maximum
- Planform: Line segment at each $\mathbf{x}$ oriented at angle $\varphi_{\mathbf{x}}$

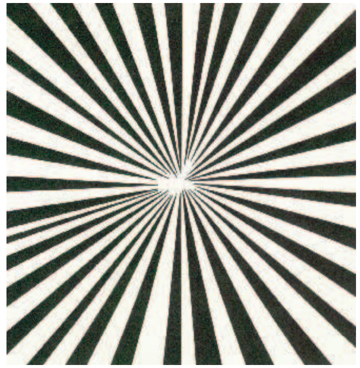
Planforms For Bressloff-Cowan



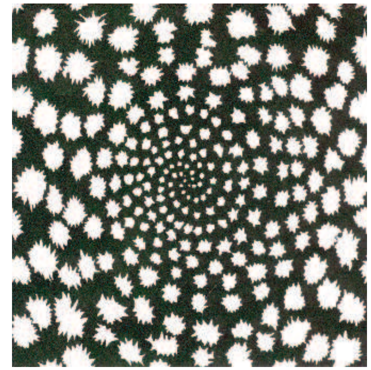
Cortex to Retina

- Neurons on cortex are uniformly distributed
- Neurons in retina fall off by $1/r^2$ from fovea
- Unique angle preserving map takes uniform density square to $1/r^2$ density disk: complex exponential
- Straight lines on cortex $\mapsto$ circles, logarithmic spirals, and rays in retina

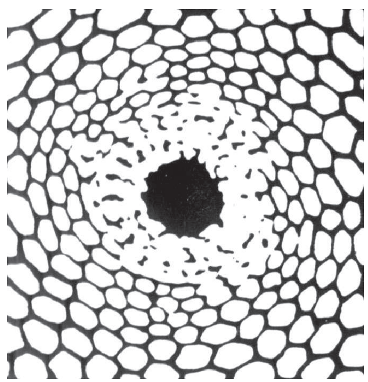
Visual Hallucinations



(I)



(II)

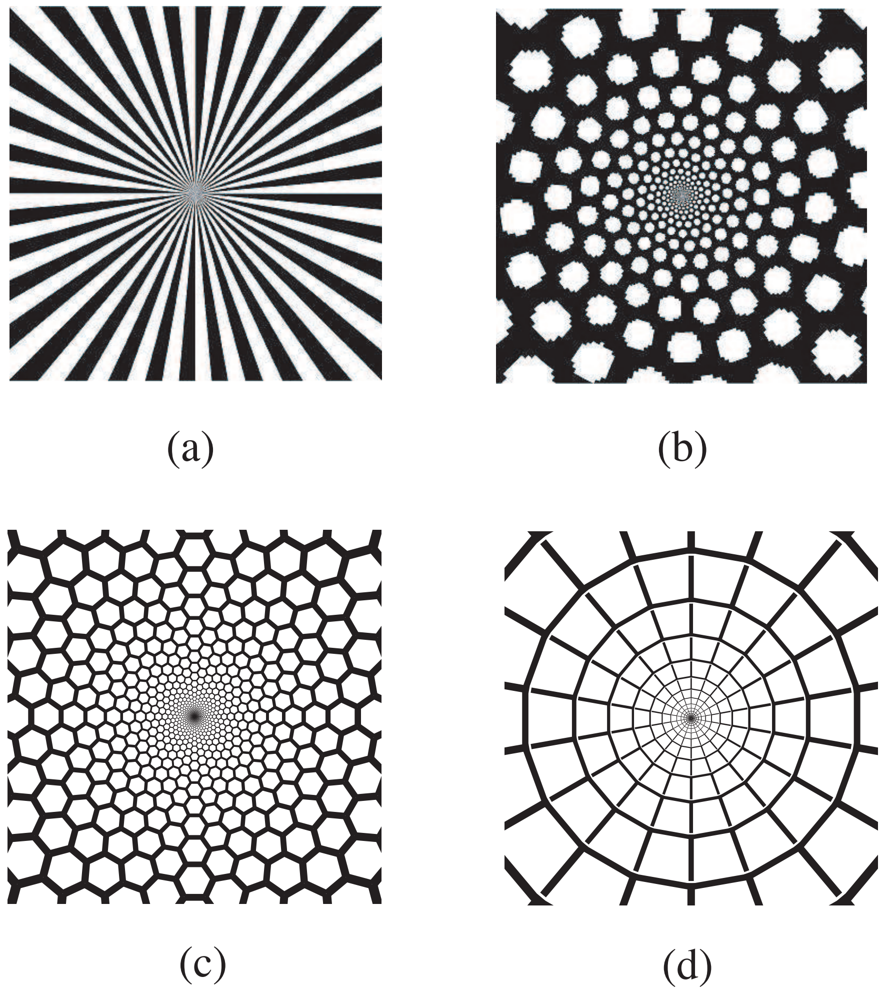


(III)

(IV)

(I) funnel and (II) spiral images LSD [Siegel & Jarvik, 1975], (III) honeycomb marihuana [Clottes & Lewis-Williams (1998)], (IV) cobweb petroglyph [Patterson, 1992]

Planforms in the Visual Field



Visual field planforms

Weakly Anisotropic Coupling

- In addition to equilibria found in Bressloff-Cowan model there exist periodic solutions that emanate from steady-state bifurcation

1. Rotating Spirals
2. Tunneling Blobs   Tunneling Spiraling Blobs
3. Pulsating Blobs

Pattern Formation Outline

1. **Bifurcation Theory with Symmetry**
   - Equivariant Branching Lemma
   - Model independent analysis

2. Translations lead to **plane waves**

3. **Planforms**: Computation of eigenfunctions

Primer on Steady-State Bifurcation

- Solve $\dot{x} = f(x, \lambda) = 0$ where $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$
- Local theory: Assume $f(0, 0) = 0$ & find solns near $(0, 0)$
- If $L = (d_x f)_{0,0}$ nonsingular, IFT implies unique soln $x(\lambda)$

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- If $L = (d_x f)_{0,0}$ nonsingular, IFT implies unique soln $x(\lambda)$
- Bifurcation of steady states $\Longleftrightarrow \ker L \neq \{0\}$
- Reduction theory implies that steady-states are found by solving $\varphi(y, \lambda) = 0$ where

$$\varphi : \ker L \times \mathbf{R} \to \ker L$$

Equivariant Steady-State Bifurcation

Let γ : $\mathbf{R}^n \to \mathbf{R}^n$ be linear

- γ is a symmetry iff γ(soln)=soln iff $f(\gamma x, \lambda) = \gamma f(x, \lambda)$
- Chain rule $\Longrightarrow L\gamma = \gamma L \Longrightarrow \ker L$ is γ-invariant
- Theorem: Fix symmetry group Γ. Generically $\ker L$ is an absolutely irreducible representation of Γ
- Reduction implies that there is a unique steady-state bifurcation theory for each absolutely irreducible rep

Equivariant Bifurcation Theory

- Let $\Sigma \subset \Gamma$ be a subgroup
- $\text{Fix}(\Sigma) = \{x \in \ker L : \sigma x = x \quad \forall \sigma \in \Sigma\}$
- Σ is *axial* if $\dim \text{Fix}(\Sigma) = 1$
- Equivariant Branching Lemma:

  Generically, there exists a branch of solutions with Σ symmetry for every axial subgroup Σ

- MODEL INDEPENDENT

  Solution types do not depend on the equation — only on the symmetry group and its representation on $\ker L$

Translations

- Let $W_{\mathbf{k}} = \{u(\varphi)e^{i\mathbf{k}\cdot\mathbf{x}} + \text{c.c.}\} \quad \mathbf{k} \in \mathbf{R}^2 =$ wave vector
- Translations act on $W_{\mathbf{k}}$ by

$$T_{\mathbf{y}}(u(\varphi)e^{i\mathbf{k}\cdot\mathbf{x}}) = u(\varphi)e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})} = \left[e^{i\mathbf{k}\cdot\mathbf{y}}u(\varphi)\right]e^{i\mathbf{k}\cdot\mathbf{x}}$$

- $L : W_{\mathbf{k}} \to W_{\mathbf{k}}$

  Eigenfunctions of L have *plane wave* factors

Reflections

- Choose **REFLECTION** ρ so that $\rho\mathbf{k} = \mathbf{k}$

$$\rho\left(u(\varphi)e^{i\mathbf{k}\cdot\mathbf{x}}\right) = \rho(u(\varphi))e^{i\mathbf{k}\cdot\mathbf{x}}$$

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- $\rho^2 = 1$ implies

$$W_{\mathbf{k}} = W_{\mathbf{k}}^{+} \oplus W_{\mathbf{k}}^{-}$$

  where ρ acts as $+1$ on $W_{\mathbf{k}}^{+}$ and -1 on $W_{\mathbf{k}}^{-}$

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- Eigenfunctions are **even** or **odd**. When $\mathbf{k} = (1, 0)$

$$\begin{aligned} u(-\varphi) &= u(\varphi) \qquad u \in W_{\mathbf{k}}^{+} \\ u(-\varphi) &= -u(\varphi) \qquad u \in W_{\mathbf{k}}^{-} \end{aligned}$$

Rotations

- Rotations act on spaces $W_{\mathbf{k}}$

$$R_\theta\left(u(\varphi)e^{i\mathbf{k}\cdot\mathbf{x}}\right) = R_\theta(u(\varphi))e^{iR_\theta(\mathbf{k})\cdot\mathbf{x}}$$

Therefore

$$R_\theta(W_{\mathbf{k}}) = W_{R_\theta(\mathbf{k})}$$

Therefore $\ker L$ is ∞-dimensional

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- **Double-periodicity**: Look for solutions on planar lattice

$$\mathcal{F}_{\mathcal{L}} = \{ f \in \mathcal{F} : f(\mathbf{x} + \boldsymbol{\ell}) = f(\mathbf{x}) \quad \forall \boldsymbol{\ell} \in \mathcal{L} \}$$

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- Finite number of rotations: $\ker$ L is finite-dimensional
- Choose lattice size so shortest dual vectors are critical

Axials in Ermentrout-Cowan Model

Name	Planform Eigenfunction
stripes	$\cos x$
squares	$\cos x + \cos y$
hexagons	$\cos(\mathbf{k}_0 \cdot \mathbf{x}) + \cos(\mathbf{k}_1 \cdot \mathbf{x}) + \cos(\mathbf{k}_2 \cdot \mathbf{x})$

$$\mathbf{k}_0 = (1, 0) \qquad \mathbf{k}_1 = \frac{1}{2}(-1, \sqrt{3}) \qquad \mathbf{k}_2 = \frac{1}{2}(-1, -\sqrt{3})$$

Axials in Bressloff-Cowan Model

Name	Planform Eigenfunction	u
squares	$u(\varphi)\cos x + u\left(\varphi - \frac{\pi}{2}\right)\cos y$	even
stripes	$u(\varphi)\cos x$	even
hexagons	$\sum_{j=0}^{2} u\left(\varphi - j\pi/3\right)\cos(\mathbf{k}_j \cdot \mathbf{x})$	even
square	$u(\varphi)\cos x - u\left(\varphi - \frac{\pi}{2}\right)\cos y$	odd
stripes	$u(\varphi)\cos x$	odd
hexagons	$\sum_{j=0}^{2} u\left(\varphi - j\pi/3\right)\cos(\mathbf{k}_j \cdot \mathbf{x})$	odd
triangles	$\sum_{j=0}^{2} u\left(\varphi - j\pi/3\right)\sin(\mathbf{k}_j \cdot \mathbf{x})$	odd
rectangles	$u\left(\varphi - \frac{\pi}{3}\right)\cos(\mathbf{k}_1 \cdot \mathbf{x}) - u\left(\varphi + \frac{\pi}{3}\right)\cos(\mathbf{k}_2 \cdot \mathbf{x})$	odd

How to Find Amplitude Function $u(\varphi)$

- Isotropic connections imply EXTRA $\mathbf{O}(2)$ symmetry
- $\mathbf{O}(2)$ decomposes $W_{\mathbf{k}}$ into sum of irreducible subspaces

$$W_{\mathbf{k},p} = \{ze^{p\varphi i}e^{i\mathbf{k}\cdot x} + \text{c.c.} : z \in \mathbf{C}\} \cong \mathbf{R}^2$$

  Eigenfunctions lie in $W_{\mathbf{k},p}$ for some p

- $W^+_{\mathbf{k},p} = \{\cos(p\varphi)e^{i\mathbf{k}\cdot x}\}$ even case

  $W^-_{\mathbf{k},p} = \{\sin(p\varphi)e^{i\mathbf{k}\cdot x}\}$ odd case

- With weak anisotropy

$$u(\varphi) \approx \cos(p\varphi) \quad \text{or} \quad u(\varphi) \approx \sin(p\varphi)$$

Rotating waves

- Suppose $\text{Fix}(\Sigma)$ is two-dimensional

  Suppose $N_\Gamma(\Sigma) = \Sigma \times \mathbf{SO}(2)$

- Then generically solutions are rotating waves of a pattern with Σ symmetry

- Leads to rotating spirals and tunnels

- Suppose $N_\Gamma(\Sigma) = \Sigma \times \mathbf{D}_4$

- Leads to pulsating solutions