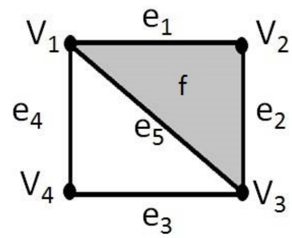
**Homework 4: Solutions**

Calculate the following for all i for the simplicial complex from class:

Figure 1: Simplicial complex from class.




1. Find $C_i$, $B_i$, $Z_i$.

   Note that $C_i$, $B_i$, $Z_i$ and $H_i$ are all vector spaces with $\mathbb{Z}_2$ coefficients. A vector space is determine by its basis. Thus one can find $C_i$, $B_i$, $Z_i$ and $H_i$ by finding bases for these vector spaces. The rank of a finite-dimensional vector space is the number of elements in a basis for that vector space.

   Recall the maps $\partial_i : C_i \to C_{i-1}$ are boundary operators.
   Recall also that $\partial_i$ is a linear map: $\partial_i(\Sigma_j n_j \sigma_j) = n_j \Sigma_j \partial_i(\sigma_j)$ where $\Sigma_j n_j \sigma_j$ is an $i$-chain = a linear combination of $i$-dimensional simplices. Thus to understand $\partial_i$, we just need to know how it acts on a simplex, $\sigma_j$.

   (a) $C_i$ = set of $i$-dimensional chains = set of all linear combination of $i$-dimensional simplices = the vector space where the basis is the set of $i$-dimensional simplices.

   - $C_0 = \mathbb{Z}_2[v_1, v_2, v_3, v_4] =< v_1, v_2, v_3, v_4 >$
   - $C_1 = \mathbb{Z}_2[e_1, e_2, e_3, e_4, e_5] =< e_1, e_2, e_3, e_4, e_5 >$
   - $C_2 = \mathbb{Z}_2[f] =< f >$
   - $C_i = \{0\}$ for all $i > 2$ because there are no $i$-dimensional simplices in this simplicial complex for $i > 2$.

   (b) $Z_i = \{\Sigma_j n_j \sigma_j \text{ in } C_i \mid \partial_i(\Sigma_j n_j \sigma_j) = 0\}$

   - Determine $Z_0$.

     Method 1: $Z_0 = \{\Sigma_i n_i v_i \text{ in } C_0 \mid \partial_0(\Sigma_i n_i v_i) = 0\} =< v_0, v_1, v_2, v_3 >= C_0$ since $\partial_0$ takes all the vertices (and thus all 0-chains) to 0.

Method 2: Using matrices. From question 2: $M_0 = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$

$Z_0 =$ null space of $M_0 =< v_1, v_2, v_3, v_4 >$

- Determine $Z_1$.

Method 1: $Z_1 = \{\Sigma_i n_i e_i \text{ in } C_1 \mid \partial_1(\Sigma_i n_i e_i) = 0\}$.

$\partial_1(\Sigma_i n_i \sigma_i) = 0$ implies $\partial_1(n_1 e_1 + n_2 e_2 + n_3 e_3 + n_4 e_4 + n_5 e_5) = 0$.

By linearity, $n_1 \partial_1(e_1) + n_2 \partial_1(e_2) + n_3 \partial_1(e_3) + n_4 \partial_1(e_4) + n_5 \partial_1(e_5) = 0$.

Thus, $n_1(v_1+v_2)+n_2(v_2+v_3)+n_3(v_3+v_4)+n_4(v_1+v_4)+n_5(v_1+v_3) = 0$

Hence, $(n_1+n_4+n_5)v_1+(n_1+n_2)v_2+(n_2+n_3+n_5)v_3+(n_3+n_4)v_4 = 0$.

This implies $n_1+n_4+n_5 = 0$, $n_1+n_2 = 0$, $n_2+n_3+n_5 = 0$, $n_3+n_4 = 0$.

Hence, $n_1 = n_2$, $n_3 = n_4$ and $n_5 = n_1 + n_4 = n_2 + n_3 = 0$.

$$\text{Thus } \Sigma_i n_i e_i \in Z_1 \text{ iff } \Sigma_i n_i e_i = n_1 e_1 + n_1 e_2 + n_4 e_3 + n_4 e_4 + (n_1 + n_4)e_5 = n_1(e_1 + e_2 + e_5) + n_4(e_3 + e_4 + e_5)$$

Thus a 1-chain is in $Z_1$ iff
it is a linear combination of the 1-chains $(e_1 + e_2 + e_5)$ and $(e_3 + e_4 + e_5)$.

Thus $\{(e_1 + e_2 + e_5), (e_3 + e_4 + e_5)\}$ is a basis for $Z_1$.

Thus, $Z_1 = \{\Sigma_i n_i e_i \text{ in } C_1 \mid \partial_1(\Sigma_i n_i e_i) = 0\}$
$= \{\Sigma_i n_i e_i \text{ in } C_1 \mid n_1 + n_4 + n_5 = 0, n_1 = n_2, n_3 = n_4\} =< e_1 + e_2 + e_5, e_3 + e_4 + e_5 >$

Method 2: Using matrices. From question 2:

$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \to \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_3 + e_4 + e_5 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\to \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_3 + e_4 + e_5 & e_1 + e_2 + e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$Z_1 = \text{null space of } M_1 =< e_1 + e_2 + e_5, e_3 + e_4 + e_5 >$

- Determine $Z_2$

  Method 1: $Z_2 = \{\Sigma_i n_i f_i \text{ in } C_2 \mid \partial_2(\Sigma_i n_i f_i) = 0\} = \{0\}$ since $\partial_2(f) = e_1 + e_2 + e_5 \neq 0$.

  Alternatively, note that $C_2 = \{0, f\}$. Since $\partial_2$ is a linear map, $\partial_2(0) = 0$. Thus $0 \in Z_2$. $\partial_2(f) = e_1 + e_2 + e_5 \neq 0$. Thus $f \notin Z_2$. Thus $Z_2 =$ kernel of $\partial_2 = \{x \text{ in } C_2 \mid \partial_2(x) = 0\} = \{0\}$

  Method 2: Using matrices. From question 2: $M_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} f \\ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$

  $Z_2 =$ null space of $M_2 = \{0\}$.

- Determine $Z_i$ for $i > 2$.

  $Z_i = \{0\}$ for all $i > 2$ since $C_i = \{0\}$ for all $i > 2$.

(c) Determine $B_i =$ image of $\partial_{i+1}$

- Determine $B_0$

  Method 1: $B_0 = \text{image of } \partial_1 = <\ \partial(e_1),\ \partial(e_2),\ \partial(e_3),\ \partial(e_4),\ \partial(e_5) >$
  $= < v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3 >$.

  Note, $v_1 + v_3 = (v_1 + v_2) + (v_2 + v_3)$.
  Thus the last generator of $< v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3 >$ is a linear combination of the first two generators. Thus $\{v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3\}$ is NOT a linearly independent set.

  Also, $v_4 + v_1 = (v_1 + v_2) + (v_2 + v_3) + (v_3 + v_4)$

  Since the last two generators are linear combinations of the first 3 generators,

$$< v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4 + v_1, v_1 + v_3 > = < v_1 + v_2, v_2 + v_3, v_3 + v_4 >$$

  Note that $\{v_1 + v_2, v_2 + v_3, v_3 + v_4\}$ is a linearly independent set and thus this set forms a basis for $B_0$.

  Thus $B_0 = < v_1 + v_2, v_2 + v_3, v_3 + v_4 >$.

Method 2: Using matrices. From question 2:

$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \to \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_1+e_2+e_3+e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\to \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_1+e_2+e_3+e_4 & e_1+e_2+e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$B_0 =$ the image of the column space of $M_1 =< v_1+v_2, v_2+v_3, v_3+v_4 >$.

- Determine $B_1$

  Method 1: $B_1 =$ image of $\partial_2 = < \partial_2(f) > = < e_1 + e_2 + e_5 >$

  Alternatively, note that $C_2 = \{0, f\}$. Since $\partial_2$ is a linear map, $\partial_2(0) = 0$. Thus $0 \in B_1$. $\partial_2(f) = e_1 + e_2 + e_5$. Thus $e_1 + e_2 + e_5 \in B_1$.

  Thus $B_1 =$ image of $\partial_2 = \{0, e_1 + e_2 + e_5\} =< e_1 + e_2 + e_5 >$

  Method 2: Using matrices. From question 2: $M_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} f \\ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$

  $B_1 =$ the image of the column space of $M_2 =< e_1 + e_2 + e_5 >$

- Determine $B_i$ for $i \geq 2$

  $B_2 =$ image of $\partial_3$ where $\partial_3 : C_3 \to C_2$. Since $C_3 = \{0\}$, $B_2 = \{0\}$.

  Similarly $B_i = \{0\}$ for $i > 2$. Thus $B_i = \{0\}$ for $i \geq 2$.

2. Find the matrix corresponding to each boundary map (from $C_i$ to $C_{i-1}$).

   (a) Let $M_0$ be the matrix corresponding to the boundary map $\partial_0 : C_0 \to 0$. Then, $\partial_0$ takes every vertex to 0.

   $$\text{Thus } M_0 = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

   (b) Let $M_1$ be the matrix corresponding to the boundary map $\partial_1 : C_1 \to C_0$. Since $\text{rank}(C_1) = 5$ and $\text{rank}(C_0) = 4$, $M_1$ is a $4 \times 5$ matrix. Also, $a_{ij}$ is nonzero iff $a_{ij} = 1$ iff the vertex corresponding to row $i$ is in the boundary of the edge corresponding to column $j$. For example, we have that $\partial(e_1) = v_1 + v_2$; hence we have a 1 as an entry in both the first and second rows of the first column. The remaining entries in the first column are zero.

   $$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

   (c) Let $M_2$ be the matrix corresponding to the boundary map $\partial_2 : C_2 \to C_1$. Since $\text{rank}(C_2) = 1$ and $\text{rank}(C_1) = 5$, $M_2$ is a $5 \times 1$ matrix. Since $\partial(f) = e_1 + e_2 + e_5$, we have a 1 as an entry in the rows corresponding to those edges.

   $$M_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{array} \begin{array}{c} f \\ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

3. Find $H_i = \frac{Z_i}{B_i}$.

   Recall that $\partial_{i+1}(\partial_i(c)) = 0$ for any c in $C_{i+1}$. Hence we get the inclusion $B_i \subset Z_i$. By modding out $Z_i$ by $B_i$, we are taking all the elements in $Z_i$ that are also in $B_i$ and setting them equal to 0.

   (a) $H_0 = Z_0/B_0 = \frac{<v_1,v_2,v_3,v_4>}{<v_1+v_2,v_2+v_3,v_3+v_4>}$

   $= < v_1, v_2, v_3, v_4 \mid v_1 + v_2 = 0, v_2 + v_3 = 0, v_3 + v_4 = 0 > = < [v_1] >$

   where $[v_1] = \{v_1, v_2, v_3, v_4\}$ is a representative of the set containing all the vertices. Since we are working with coefficients in $\mathbb{Z}_2$, we get that $\{v_1 + v_2 = 0, v_2 + v_3 = 0, v_3 + v_4 = 0\}$ implies $v_1 = v_2$, $v_2 = v_3$, $v_3 = v_4$. Thus $[v_1] = \{v_1, v_2, v_3, v_4\}$

   Rank $H_0$ = Rank $Z_0$ - Rank $B_0 = 4 - 3 = 1$

   (b) $H_1 = Z_1/B_1 = \frac{<e_1+e_2+e_5,e_3+e_4+e_5>}{<e_1+e_2+e_5>}$

   $= < e_1 + e_2 + e_5, e_3 + e_4 + e_5 \mid e_1 + e_2 + e_5 = 0 > = < [e_3 + e_4 + e_5] >$

   The cycle $e_1 + e_2 + e_5$ was filled by the face $f$ and thus $e_1 + e_2 + e_5 = 0$ in $H_1$.

   Rank $H_1$ = Rank $Z_1$ - Rank $B_1 = 2 - 1 = 1$

   Sidenote: To determine $H_1$, we set every element in $B_1$ equal to 0.
   Thus $\partial(f) = e_1 + e_2 + e_5 = 0$. Thus in $H_1$,
   $e_3 + e_4 + e_5 = (e_3 + e_4 + e_5) + 0 = (e_3 + e_4 + e_5) + \partial(f) = (e_3 + e_4 + e_5) + (e_1 + e_2 + e_5) = e_1 + e_2 + e_3 + e_4$.

   Thus $e_1 + e_2 + e_3 + e_4 \in [e_3 + e_4 + e_5]$ in $H_1$. This also means that $[e_1 + e_2 + e_3 + e_4] = [e_3 + e_4 + e_5]$ and thus $H_1 =< e_3 + e_4 + e_5 >=< e_1 + e_2 + e_3 + e_4 >$.

   Hence $H_1 =< e_3 + e_4 + e_5 >$ and $H_1 =< e_1 + e_2 + e_3 + e_4 >$ are both correct answers.

   (c) $H_2 = Z_2/B_2 = < 0 >$ since $Z_2 = < 0 >$.

   (d) Similarly, $H_i = < 0 >$ for all $i > 2$.