The following slides heavily depend on [Chazal et al., 2016, Bubenik and Scott, 2014]

Plus a couple of examples from [Ghrist, 2014, Kleinberg, 2002, Carlsson and Mémoli, 2010]

Section number are from [Chazal et al., 2016]

Category latex comes from [Baez, 2004]

Defn [Baez, 2004]: A **category** C consists of:

- a collection $\mathrm{Ob}(C)$ of **objects**.
- for any pair of objects x, y, a set of **morphisms** from x to y, written $f : x \to y$.

equipped with:

- for any object x, an **identity morphism** $1_x : x \to x$.
- for any pair of morphisms $f : x \to y$ and $g : y \to z$, a morphism $fg : x \to z$ called the **composite** of f and g.

such that:

- for any morphism $f : x \to y$, the **left and right unit laws** hold: $1_x f = f = f 1_y$.
- for any triple of morphisms $f : w \to x$, $g : x \to y$, $h : y \to z$, the **associative law** holds: $(fg)h = f(gh)$.

Example: **FinMet**$^{\leq}$

Objects: (X, d_X) = finite metric space

Morphisms: $f : (X, d_X) \to (Y, d_Y)$ s. t.
$d_Y(f(x), f(y)) \leq d_X(x, y)$

Example: **Clust**

Objects:
$(X, \mathcal{P}(X))$ where X is a finite set and $\mathcal{P}(X)$ is a partition of X.

Note: elements of $\mathcal{P}(X)$ are called clusters.

Morphisms:
$f : (X, \mathcal{P}(X)) \to (Y, \mathcal{P}(Y))$ s.t. $\mathcal{P}(X)$ is a refinement of $f^{-1}(\mathcal{P}(Y))$.

Note: a cluster morphism can coalesce clusters, but not break them up

Defn [Baez, 2004]: Given categories C, D, a **functor** $F : C \to D$ consists of:

- a function $F : \mathrm{Ob}(C) \to \mathrm{Ob}(D)$.
- for any pair of objects $x, y \in \mathrm{Ob}(C)$, a function $F : \mathit{morphism}(x \to y) \to \mathit{morphism}(F(x) \to F(y))$.

such that:

- F **preserves identities**: for any object $x \in C$, $F(1_x) = 1_{F(x)}$.
- F **preserves composition**: for any pair of morphisms $f : x \to y$, $g : y \to z$ in C, $F(fg) = F(f)F(g)$.

Kleinberg's Impossibility Theorem [Kleinberg, 2002]: There is no nontrivial functor from $\mathrm{FinMet}^{\leq}$ **onto** Clust

Kleinberg's clustering axioms:

Scale-Invariance. For any distance function d and any $\alpha > 0$, we have $F(d) = F(\alpha \cdot d)$.

Richness. Range(F) is equal to the set of all partitions of S. Richness requires that for any desired partition $\mathcal{P}$, it should be possible to construct a distance function d on S for which $F(d) = \mathcal{P}$.

Consistency. Let d and d_0 be two distance functions. If $F(d) = \mathcal{P}$, and d_0 is a $\mathcal{P}$-transformation of d, then $F(d_0) = \mathcal{P}$.

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Let $\mathcal{P}$ be a partition of S, and d and d_0 two distance functions on S. We say that d_0 is a $\mathcal{P}$-transformation of d if

(a) for all i, j belonging to the same cluster of $\mathcal{P}$, we have $d_0(i, j) \leq d(i, j)$; and

(b) for all i, j belonging to different clusters of $\mathcal{P}$, we have $d_0(i, j) \geq d(i, j)$.

In other words, suppose that the clustering $\mathcal{P}$ arises from the distance function d. If we now produce d_0 by reducing distances within the clusters and enlarging distance between the clusters then the same clustering $\mathcal{P}$ should arise from d_0.

Kleinberg's clustering axioms:

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Kleinberg's Impossibility Theorem [Kleinberg, 2002]: There is no nontrivial functor from $\mathrm{FinMet}^{\leq}$ **onto** Clust

Example: **PClust**

Objects:
$(X, \mathcal{P}_t(X))$ where X is a finite set and $\mathcal{P}_t(X)$ is a famiy of partitions of X such that $\mathcal{P}_t(X)$ is a refinement of $\mathcal{P}_s(X)$ if $t \leq s$.

Note: $\mathcal{P}_t(X)$ can be represented by a dendrogram.

Morphisms:
$f : (X, \mathcal{P}_t(X)) \to (Y, \mathcal{P}'_t(Y))$ s.t. $\mathcal{P}_t(X)$ is a refinement of $f^{-1}(\mathcal{P}'_t(Y))$.

Thm: [Carlsson and Mémoli, 2010]

$\exists!$ functor $\mathbf{FinMet}^{\leq} \to \mathbf{PClust}$ that takes the input $X = \{a, b\}$ where $d(a, b) = R$ to

$$\mathcal{P}_t(X) = \{a\}, \{b\} \text{ for } t < R \text{ and } \mathcal{P}_t(X) = \{a, b\} \text{ for } t \geq R.$$

The output corresponds to single linkage clustering.

2.1 Persistence Modules Over a Real Parameter [Chazal et al., 2016]

Example: **Real line**

Objects: real numbers

Morphisms: $s \to t$ if $s \leq t$

Example: **Vec**

Objects: Vector spaces

Morphisms: Linear maps

Example: Persistence module $\mathbb{V}$ over $\mathbb{R}$ is a functor from the **Real line** into **Vec**

I.e., $\mathbb{V} = \{V_t \mid t \in \mathbb{R}\}$ with linear maps $\{v_t^s : V_s \to V_t \mid s \leq t\}$

Example: **Set**

Objects: Sets

Morphisms: subset relation

(Closed) sublevelset filtration of $(X, f) = \mathbb{X}_{sub} = \mathbb{X}^f_{sub}$ is a functor from the **Real line** into **Set**.

Let $f : X \to \mathbb{R}$

Let $X^t = (X, f)^t = \{x \in X \mid f(x) \leq t\} = f^{-1}(\infty, t]$

Example: **Top**

Objects: Topological spaces.

Morphisms: continuous maps.

H_n is a functor from **Top** into **Vec**:

$V_t = H(X^t)$, $v_s^t = H(i_t^s) : V_s \to V_t$ is a persistent module.

$\mathbb{V}$ is **q-tame** if $r_t^s = rank(v_t^s) \leq \infty$ whenever $s < t$

Example: **grVec**

Objects: Graded vector spaces

Morphisms: Linear maps $f : V_n \to W_n$ where both vector spaces have the same grade n.

H_* is a functor from **Top** into **grVec**

2.2 Index Posets [Chazal et al., 2016]

A $(\mathbf{T}, \leq)$ is partially ordered if $\leq$ is reflexive, anti-symmetric, and transitive. Then $\mathbf{T}$ is a category with morphisms $\leq$.

$\mathbf{T}$-Persistence module $\mathbb{V}$ is a functor from $\mathbf{T}$ into Vec

I.e., $\mathbb{V} = \{V_t \mid t \in \mathbf{T}\}$ with linear maps $\{v_t^s : V_s \to V_t \mid s \leq t\}$

If $\mathbf{S} \subset \mathbf{T}$, then $\mathbb{V}_{\mathbf{S}} = \mathbb{V}|_{\mathbf{S}} =$ the **restriction** of $\mathbb{V}$ to $\mathbf{S}$.

Example $\{1, ..., m\} \subset \mathbb{R}$.

$\{1, ..., m\}$-Persistence module $\mathbb{V}_m$ is the restriction of the persistence module $\mathbb{V}$ over $\mathbb{R}$

Defn: Given functors $F, G : C \to D$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of:

- a function α mapping each object $x \in C$ to a morphism $\alpha_x : F(x) \to G(x)$

such that:

- for any morphism $f : x \to y$ in C, this diagram commutes:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

Defn: Given **T**-persistent modules $\mathbb{U}, \mathbb{V} : \mathbf{T} \to \mathbf{Vec}$, a **homomorphism** $\phi : \mathbb{U} \Rightarrow \mathbb{V}$ consists of:

- a collection of linear maps $\{\phi_t : U_t \to V_t \mid t \in \mathbf{T}\}$

such that:

- for any morphism $s \leq t$ in $\mathbb{U}$, this diagram commutes:

$$\begin{array}{ccc} U_s & \xrightarrow{u_t^s} & U_t \\ {\scriptstyle \phi_s}\downarrow & & \downarrow{\scriptstyle \phi_t} \\ V_s & \xrightarrow[v_t^s]{} & V_t \end{array}$$

$$Hom(\mathbb{U}, \mathbb{V}) = \{homomorphisms\ \mathbb{U} \Rightarrow \mathbb{V}\}$$

$$End(\mathbb{V}) = \{homomorphisms\ \mathbb{V} \Rightarrow \mathbb{V}\}$$

2.4 Interval Modules [Chazal et al., 2016]

Let $\mathbf{T}$ be a totally ordered set.

$J \subset \mathbf{T}$ is an **interval** if $r, t \in J$ and if $r < s < t$, then $s \in J$.

The **interval module** $\mathbb{I} = \mathbf{k}^J$ is the $\mathbf{T}$-persistence module with vector spaces

$$I_t = \begin{cases} \mathbf{k} & \textit{if } \ t \in J \\ 0 & \textit{otherwise} \end{cases}$$

and linear maps

$$i_t^s = \begin{cases} 1 & \textit{if } \ s, t \in J \\ 0 & \textit{otherwise} \end{cases}$$

In informal language, $\mathbf{k}^J$ represents a 'feature' which 'persists' over exactly the interval J and nowhere else.

I.e, $\mathbf{k}^J$ represents a bar in the barcode.

2.5 Interval Decomposition [Chazal et al., 2016]

The **direct sum** $\mathbb{W} = \mathbb{U} \oplus \mathbb{V}$ of two persistence modules $\mathbb{U}$, $\mathbb{V}$ is the category with

Objects: $W_t = U_t \oplus V_t$

Morphisms: $w_t^s = u_t^s \oplus v_t^s$.

A persistence module $\mathbb{W}$ is **indecomposible** if

$$\mathbb{W} = \mathbb{U} \oplus \mathbb{V} \text{ implies } \mathbb{U}, \mathbb{V} \in \{0, \mathbb{W}\}$$

Given an indexed family of intervals $\{J_\ell \mid \ell \in L\}$ we can synthesize a persistence module $\mathbb{V} = \bigoplus_{\ell \in L} \mathbf{k}^{J_\ell}$ whose isomorphism type depends only on the multiset $\{J_\ell \mid \ell \in L\}$.

Given a persistence module, $\mathbb{V}$, we can often decompose $\mathbb{V}$ into submodules isomorphic to interval modules.

The decomposition of a persistence module is frequently described in metaphorical terms. The index $t \in \mathbb{R}$ is interpreted as time. Each interval summand $\mathbf{k}^J$ represents a feature of the module which is born at time $inf(J)$ and dies at time $sup(J)$.

Theorem 2.8 (Gabriel, Auslander,RingelTachikawa,Webb, Crawley-Boevey) Let $\mathbb{V}$ be a persistence module over $\mathbf{T} \subset \mathbb{R}$. Then $\mathbb{V}$ can be decomposed as a direct sum of interval modules in either of the following situations:

(1) $\mathbf{T}$ is a finite set; or
(2) each V_t is finite-dimensional.

On the other hand, there exists a persistence module over $\mathbb{Z}$ (indeed, over the nonpositive integers) which does not admit an interval decomposition.

Prop 2.5 Let $\mathbb{I} = \mathbf{k_T}^J$ be an interval module over $\mathbf{T} \subset \mathbb{R}$, then $End(\mathbb{I}) = \mathbb{R}$.

Prop 2.6: Interval modules are indecomposible.

2.6 The Decomposition Persistence Diagram [Chazal et al., 2016]

Let $\mathbf{k}(p^*, q^*) = \mathbf{k}^{(p^*, q^*)}$ where (p^*, q^*) represents an interval (open, closed, or half-open).

If a persistence module $\mathbb{V}$ indexed over $\mathbb{R}$ can be decomposed,

$$\mathbb{V} \cong \bigoplus_{\ell \in L} \mathbf{k}((p_\ell^*, q\ell^*)$$

Then we define the **decorated persistence diagram** to the be multiset:

$$Dgm(\mathbb{V}) = Int(\mathbb{V}) = \{(p_\ell^*, q\ell^*) \mid \ell \in L\}$$

and the **undecorated persistence diagram** to the be multiset:

$$dgm(\mathbb{V}) = int(\mathbb{V}) = \{(p_\ell^*, q\ell^*) \mid \ell \in L\} - \Delta$$

where $\Delta = \{(r, r) \mid r \in \mathbb{R}\}$ = the diagonal.

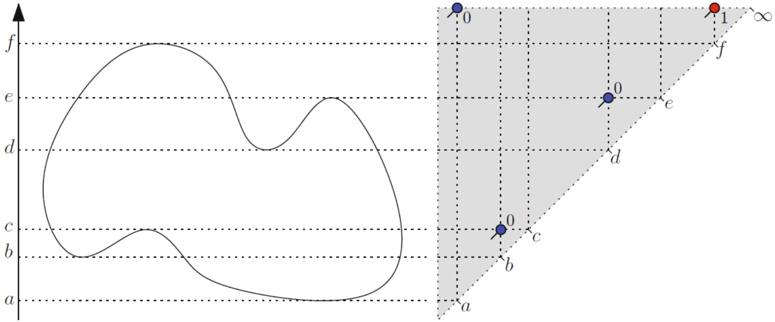

Fig. 2.3 A traditional example. *Left:* X is a smoothly embedded curve in the plane, and f is its y-coordinate or 'height' function. *Right:* The decorated persistence diagram of $\mathrm{H}(\mathbb{X}_{\mathrm{sub}})$: there are three intervals in H_0 (*blue dots*, marked 0) and one interval in H_1 (*red dot*, marked 1)

A persistence module $\mathbb{V}$ indexed over a finite subset of the real line

$$\mathbf{T} : a_1 < a_2 < \cdot < a_n$$

can be thought of as a diagram of n vector spaces and $n-1$ linear maps: $\mathbb{V} : V_{a_1} \to V_{a_1} \to \ \to V_{a_n}$

Such a diagram can be represented by a quiver (multidigraph):

Example 2.13 Let $a < b < c$. There are six interval modules over $\{a, b, c\}$, namely:

$\mathbf{k}[a, a] = \bullet_a \text{———} \circ_b \text{———} \circ_c$ $\quad$ $\mathbf{k}[a, b] = \bullet_a \text{———} \bullet_b \text{———} \circ_c$

$\mathbf{k}[b, b] = \circ_a \text{———} \bullet_b \text{———} \circ_c$ $\quad$ $\mathbf{k}[b, c] = \circ_a \text{———} \bullet_b \text{———} \bullet_c$

$\mathbf{k}[c, c] = \circ_a \text{———} \circ_b \text{———} \bullet_c$ $\quad$ $\mathbf{k}[a, c] = \bullet_a \text{———} \bullet_b \text{———} \bullet_c$

If $\mathbf{k}[a,b] = \bullet_a \!\!-\!\!-\!\!-\!\!-\!\!\bullet_b \!\!-\!\!-\!\!-\!\!-\!\!\circ_c$ occurs with multiplicity m in the interval decomposition of $\mathbb{V}$, then

$$m \;=\; \langle\, [a,b] \mid \mathbb{V}_{a,b,c} \rangle \;=\; \langle \bullet_a \!\!-\!\!-\!\!-\!\!-\!\!\bullet_b \!\!-\!\!-\!\!-\!\!-\!\!\circ_c \rangle$$

Example 2.15 The invariants of a single linear map $v : V_a \to V_b$ are:

$$rank(v) = \langle \bullet_a \text{———} \bullet_b \rangle$$

$$nullity(v) = \langle \bullet_a \text{———} \circ_b \rangle$$

$$conullity(v) = \langle \circ_a \text{———} \bullet_b \rangle$$

Example: If $a \leq b \leq c \leq d$, then
$rank(V_b \to V_c) \geq rank(V_a \to V_d)$

Proof:

$$
\begin{aligned}
rank(V_b \to V_c) &= \langle \text{------} \bullet_b \text{------} \bullet_c \text{------} \rangle \\
&= \langle \bullet_a \text{------} \bullet_b \text{------} \bullet_c \text{------} \bullet_d \rangle \\
&\quad + \langle \circ_a \text{------} \bullet_b \text{------} \bullet_c \text{------} \bullet_d \rangle \\
&\quad + \langle \bullet_a \text{------} \bullet_b \text{------} \bullet_c \text{------} \circ_d \rangle \\
&\quad + \langle \circ_a \text{------} \bullet_b \text{------} \bullet_c \text{------} \circ_d \rangle \\
&\geq \langle \bullet_a \text{------} \bullet_b \text{------} \bullet_c \text{------} \bullet_d \rangle \\
&= \langle \bullet_a \text{------------------} \bullet_d \rangle \\
&= rank(V_a \to V_d)
\end{aligned}
$$

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