The following slides heavily depend on [Chazal et al., 2016, Bubenik and Scott, 2014]


Section number are from [Chazal et al., 2016]

Category latex comes from [Baez, 2004]
Defn [Baez, 2004]: A **category** $C$ consists of:

- a collection $\text{Ob}(C)$ of **objects**.
- for any pair of objects $x, y$, a set of **morphisms** from $x$ to $y$, written $f : x \to y$.

equipped with:

- for any object $x$, an **identity morphism** $1_x : x \to x$.
- for any pair of morphisms $f : x \to y$ and $g : y \to z$, a morphism $fg : x \to z$ called the **composite** of $f$ and $g$.

such that:

- for any morphism $f : x \to y$, the **left and right unit laws** hold: $1_x f = f = f 1_y$.
- for any triple of morphisms $f : w \to x$, $g : x \to y$, $h : y \to z$, the **associative law** holds: $(fg)h = f(gh)$. 
Example: \textbf{FinMet} ≤

Objects: \((X, d_X) = \text{finite metric space}\)

Morphisms: \(f : (X, d_X) \rightarrow (Y, d_Y)\) s.t. 
\[d_Y(f(x), f(y)) \leq d_X(x, y)\]

Example: \textbf{Clust}

Objects: 
\((X, \mathcal{P}(X))\) where \(X\) is a finite set and \(\mathcal{P}(X)\) is a partition of \(X\).

Note: elements of \(\mathcal{P}(X)\) are called clusters.

Morphisms: 
\(f : (X, \mathcal{P}(X)) \rightarrow (Y, \mathcal{P}(Y))\) s.t. \(\mathcal{P}(X)\) is a refinement of \(f^{-1}(\mathcal{P}(Y))\).

Note: a cluster morphism can coalesce clusters, but not break them up.
Defn [Baez, 2004]: Given categories $C, D$, a **functor** $F : C \to D$ consists of:

- a function $F : \text{Ob}(C) \to \text{Ob}(D)$.
- for any pair of objects $x, y \in \text{Ob}(C)$, a function $F : \text{morphism}(x \to y) \to \text{morphism}(F(x) \to F(y))$.

such that:

- **$F$ preserves identities**: for any object $x \in C$, $F(1_x) = 1_{F(x)}$.
- **$F$ preserves composition**: for any pair of morphisms $f : x \to y$, $g : y \to z$ in $C$, $F(fg) = F(f)F(g)$.
Kleinberg’s Impossibility Theorem [Kleinberg, 2002]: There is no nontrivial functor from \( \text{FinMet} \leq \) onto \( \text{Clust} \)
Kleinberg’s clustering axioms:

Scale-Invariance. For any distance function \( d \) and any \( \alpha > 0 \), we have \( F(d) = F(\alpha \cdot d) \).

Richness. \( \text{Range}(F) \) is equal to the set of all partitions of \( S \). Richness requires that for any desired partition \( \mathcal{P} \), it should be possible to construct a distance function \( d \) on \( S \) for which \( F(d) = \mathcal{P} \).

Consistency. Let \( d \) and \( d_0 \) be two distance functions. If \( F(d) = \mathcal{P} \), and \( d_0 \) is a \( \mathcal{P} \)-transformation of \( d \), then \( F(d_0) = \mathcal{P} \).
Consistency. Let \( d \) and \( d_0 \) be two distance functions. If \( F(d) = \mathcal{P} \), and \( d_0 \) is a \( \mathcal{P} \)-transformation of \( d \), then \( F(d_0) = \mathcal{P} \).

Let \( \mathcal{P} \) be a partition of \( S \), and \( d \) and \( d_0 \) two distance functions on \( S \). We say that \( d_0 \) is a \( \mathcal{P} \)-transformation of \( d \) if

(a) for all \( i, j \) belonging to the same cluster of \( \mathcal{P} \), we have \( d_0(i, j) \leq d(i, j) \); and

(b) for all \( i, j \) belonging to different clusters of \( \mathcal{P} \), we have \( d_0(i, j) \geq d(i, j) \).

In other words, suppose that the clustering \( \mathcal{P} \) arises from the distance function \( d \). If we now produce \( d_0 \) by reducing distances within the clusters and enlarging distance between the clusters then the same clustering \( \mathcal{P} \) should arise from \( d_0 \).
Kleinberg’s clustering axioms:

Scale-Invariance. For any distance function \( d \) and any \( \alpha > 0 \), we have \( f(d) = f(\alpha \cdot d) \).

Richness. Range(\( f \)) is equal to the set of all partitions of \( S \). Richness requires that for any desired partition \( \mathcal{P} \), it should be possible to construct a distance function \( d \) on \( S \) for which \( f(d) = \mathcal{P} \).

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Kleinberg’s Impossibility Theorem [Kleinberg, 2002]: There is no nontrivial functor from FinMet\( \leq \) onto Clust
Example: \textbf{PClust}

Objects:
$(X, \mathcal{P}_t(X))$ where $X$ is a finite set and $\mathcal{P}_t(X)$ is a family of partitions of $X$ such that $\mathcal{P}_t(X)$ is a refinement of $\mathcal{P}_s(X)$ if $t \leq s$. 

Note: $\mathcal{P}_t(X)$ can be represented by a dendrogram.

Morphisms:
$f : (X, \mathcal{P}_t(X)) \to (Y, \mathcal{P}'_t(Y))$ s.t. $\mathcal{P}_t(X)$ is a refinement of $f^{-1}(\mathcal{P}'_t(Y))$.

Thm: [Carlsson and Mémoli, 2010]

$\exists$! functor $\textbf{FinMet}^\leq \to \textbf{PClust}$ that takes the input $X = \{a, b\}$ where $d(a, b) = R$ to

$\mathcal{P}_t(X) = \{a\}, \{b\}$ for $t < R$ and $\mathcal{P}_t(X) = \{a, b\}$ for $t \geq R$.

The output corresponds to single linkage clustering.
2.1 Persistence Modules Over a Real Parameter [Chazal et al., 2016]

Example: **Real line**

Objects: real numbers

Morphisms: \( s \rightarrow t \) if \( s \leq t \)

Example: **Vec**

Objects: Vector spaces

Morphisms: Linear maps

Example: Persistence module \( V \) over \( \mathbb{R} \) is a functor from the **Real line** into **Vec**

i.e., \( V = \{ V_t \mid t \in \mathbb{R} \} \) with linear maps \( \{ v_t^s : V_s \rightarrow V_t \mid s \leq t \} \)
Example: **Set**

Objects: Sets

Morphisms: subset relation

(Closed) sublevelset filtration of \((X, f) = X_{sub} = X^f_{sub}\) is a functor from the **Real line** into **Set**.

Let \(f : X \to \mathbb{R}\)

Let \(X^t = (X, f)^t = \{x \in X \mid f(x) \leq t\} = f^{-1}(\infty, t]\)
Example: **Top**

   Objects: Topological spaces.

   Morphisms: continuous maps.

$H_n$ is a functor from **Top** into **Vec**:

$V_t = H(X^t), \ v^t_s = H(i^s_t) : V_s \rightarrow V_t$ is a persistent module.

\[ \forall \text{ is q-tame if } r^s_t = \text{rank}(v^s_t) \leq \infty \text{ whenever } s < t \]

Example: **grVec**

   Objects: Graded vector spaces

   Morphisms: Linear maps $f : V_n \rightarrow W_n$ where both vector spaces have the same grade $n$.

$H_\ast$ is a functor from **Top** into **grVec**
A \( (\mathcal{T}, \leq) \) is partially ordered if \( \leq \) is reflexive, anti-symmetric, and transitive. Then \( \mathcal{T} \) is a category with morphisms \( \leq \).

\( \mathcal{T} \)-Persistence module \( \mathbb{V} \) is a functor from \( \mathcal{T} \) into Vec

i.e., \( \mathbb{V} = \{ V_t \mid t \in \mathcal{T} \} \) with linear maps \( \{ v_t^s : V_s \to V_t \mid s \leq t \} \)

If \( S \subset \mathcal{T} \), then \( \mathbb{V}_S = \mathbb{V}|_S \) = the restriction of \( \mathbb{V} \) to \( S \).

Example \( \{1, \ldots, m\} \subset \mathbb{R} \).

\( \{1, \ldots, m\} \)-Persistence module \( \mathbb{V}_m \) is the restriction of the persistence module \( \mathbb{V} \) over \( \mathbb{R} \)
Defn: Given functors $F, G : C \to D$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of:

- a function $\alpha$ mapping each object $x \in C$ to a morphism $\alpha_x : F(x) \to G(x)$

such that:

- for any morphism $f : x \to y$ in $C$, this diagram commutes:

$$
\begin{align*}
F(x) & \xrightarrow{F(f)} F(y) \\
\downarrow_{\alpha_x} & \downarrow_{\alpha_y} \\
G(x) & \xrightarrow{G(f)} G(y)
\end{align*}
$$
Defn: Given $\mathbf{T}$-persistent modules $U, V : \mathbf{T} \to \mathbf{Vec}$, a homomorphism $\phi : U \Rightarrow V$ consists of:

- a collection of linear maps $\{\phi_t : U_t \to V_t \mid t \in \mathbf{T}\}$

such that:

- for any morphism $s \leq t$ in $U$, this diagram commutes:

\[
\begin{array}{ccc}
U_s & \xrightarrow{u_t^s} & U_t \\
\downarrow \phi_s & & \downarrow \phi_t \\
V_s & \xrightarrow{v_t^s} & V_t
\end{array}
\]

$\text{Hom}(U, V) = \{\text{homomorphisms } U \Rightarrow V\}$

$\text{End}(V) = \{\text{homomorphisms } V \Rightarrow V\}$
2.4 Interval Modules [Chazal et al., 2016]

Let $T$ be a totally ordered set.

$J \subset T$ is an **interval** if $r, t \in J$ and if $r < s < t$, then $s \in J$.

The **interval module** $I = k^J$ is the $T$-persistence module with vector spaces

$$I_t = \begin{cases} k & \text{if } t \in J \\ 0 & \text{otherwise} \end{cases}$$

and linear maps

$$i_{t}^s = \begin{cases} 1 & \text{if } s, t \in J \\ 0 & \text{otherwise} \end{cases}$$

In informal language, $k^J$ represents a 'feature' which 'persists' over exactly the interval $J$ and nowhere else.

I.e, $k^J$ represents a bar in the barcode.
The **direct sum** $W = U \oplus V$ of two persistence modules $U$, $V$ is the category with

- **Objects:** $W_t = U_t \oplus V_t$

- **Morphisms:** $w_t^s = u_t^s \oplus v_t^s$.

A persistence module $W$ is **indecomposable** if

$$W = U \oplus V \text{ implies } U, V \in \{0, W\}$$

Given an indexed family of intervals $\{J_\ell \mid \ell \in L\}$ we can synthesize a persistence module $V = \bigoplus_{\ell \in L} k^{J_\ell}$ whose isomorphism type depends only on the multiset $\{J_\ell \mid \ell \in L\}$.

Given a persistence module, $V$, we can often decompose $V$ into submodules isomorphic to interval modules.
The decomposition of a persistence module is frequently described in metaphorical terms. The index $t \in \mathbb{R}$ is interpreted as time. Each interval summand $k^J$ represents a feature of the module which is born at time $\inf(J)$ and dies at time $\sup(J)$. 
Theorem 2.8 (Gabriel, Auslander, Ringel, Tachikawa, Webb, Crawley-Boevey) Let $\mathcal{V}$ be a persistence module over $T \subset \mathbb{R}$. Then $\mathcal{V}$ can be decomposed as a direct sum of interval modules in either of the following situations:

(1) $T$ is a finite set; or
(2) each $V_t$ is finite-dimensional.

On the other hand, there exists a persistence module over $\mathbb{Z}$ (indeed, over the nonpositive integers) which does not admit an interval decomposition.
Prop 2.5 Let $\mathbb{I} = k_T^J$ be an interval module over $T \subset \mathbb{R}$, then $End(\mathbb{I}) = \mathbb{R}$.

Prop 2.6: Interval modules are indecomposable.
2.6 The Decomposition Persistence Diagram
[Chazal et al., 2016]

Let $k(p^*, q^*) = k(p^*, q^*)$ where $(p^*, q^*)$ represents an interval (open, closed, or half-open).

If a persistence module $\mathbb{V}$ indexed over $\mathbb{R}$ can be decomposed,

$$\mathbb{V} \cong \bigoplus_{\ell \in L} k((p^*_\ell, q^*_\ell))$$

Then we define the **decorated persistence diagram** to be the multiset:

$$Dgm(\mathbb{V}) = Int(\mathbb{V}) = \{(p^*_\ell, q^*_\ell) \mid \ell \in L\}$$

and the **undecorated persistence diagram** to be the multiset:

$$dgm(\mathbb{V}) = int(\mathbb{V}) = \{(p^*_\ell, q^*_\ell) \mid \ell \in L\} - \Delta$$

where $\Delta = \{(r, r) \mid r \in \mathbb{R}\} = \text{the diagonal.}$
Fig. 2.3  A traditional example. *Left:* $X$ is a smoothly embedded curve in the plane, and $f$ is its y-coordinate or ‘height’ function. *Right:* The decorated persistence diagram of $H(X_{\text{sub}})$: there are three intervals in $H_0$ (*blue dots*, marked 0) and one interval in $H_1$ (*red dot*, marked 1).
A persistence module $\mathcal{V}$ indexed over a finite subset of the real line $T: a_1 < a_2 < \cdots < a_n$ can be thought of as a diagram of $n$ vector spaces and $n - 1$ linear maps: $\mathcal{V}: V_{a_1} \to V_{a_2} \to \cdots \to V_{a_n}$.

Such a diagram can be represented by a quiver (multidigraph):

Example 2.13 Let $a < b < c$. There are six interval modules over $\{a, b, c\}$, namely:

- $k[a, a] = \bullet_a \circlearrowright \circ_b \circlearrowright \circ_c$
- $k[a, b] = \bullet_a \circlearrowright \bullet_b \circlearrowright \circ_c$
- $k[b, b] = \circ_a \bullet_b \circlearrowright \circ_c$
- $k[b, c] = \circ_a \bullet_b \bullet_c$
- $k[c, c] = \circ_a \circ_b \bullet_c$
- $k[a, c] = \bullet_a \bullet_b \bullet_c$
If $k[a, b] = \bullet_a \overline{\overline{\overline{\bullet b \circlearrowleft c}}}$ occurs with multiplicity $m$ in the interval decomposition of $\nabla$, then

$$m = \langle [a, b] \mid \nabla_{a, b, c} \rangle = \langle \bullet_a \overline{\overline{\overline{\bullet b \circlearrowleft c}}}_c \rangle$$
Example 2.15 The invariants of a single linear map $v : V_a \to V_b$ are:

$$\text{rank}(v) = \langle \bullet_a \longrightarrow \bullet_b \rangle$$

$$\text{nullity}(v) = \langle \bullet_a \longrightarrow \circ_b \rangle$$

$$\text{conullity}(v) = \langle \circ_a \longrightarrow \bullet_b \rangle$$
Example: If \( a \leq b \leq c \leq d \), then
\[
\text{rank}(V_b \to V_c) \geq \text{rank}(V_a \to V_d)
\]

Proof:
\[
\text{rank}(V_b \to V_c) = \langle \bullet_b \bullet_c \rangle
\]
\[
= \langle \bullet_a \bullet_b \bullet_c \bullet_d \rangle
\]
\[
+ \langle \circ_a \bullet_b \bullet_c \bullet_d \rangle
\]
\[
+ \langle \bullet_a \bullet_b \bullet_c \circ_d \rangle
\]
\[
+ \langle \circ_a \bullet_b \bullet_c \circ_d \rangle
\]
\[
\geq \langle \bullet_a \bullet_b \bullet_c \bullet_d \rangle
\]
\[
= \langle \bullet_a \bullet_d \rangle
\]
\[
= \text{rank}(V_a \to V_d)
\]
References I

Some definitions everyone should know.

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