Homework 5: Solutions

Let $C$ be the simplicial complex below (the boundary of a tetrahedron).

Find the following:

- $C_0 = \mathbb{Z}_2[ v_1, v_2, v_3, v_4 ] = < v_1, v_2, v_3, v_4 >$
- $C_2 = \mathbb{Z}_2[ f_1, f_2, f_3, f_4 ] = < f_1, f_2, f_3, f_4 >$
- $C_3 = < 0 >$

- $Z_0 = \{ \sum_i n_i v_i \text{ in } C_0 \mid \delta_0(\sum_i n_i v_i) = 0 \} = < v_1, v_2, v_3, v_4 > = C_0$

  Explain your answer for $Z_0$: The elements in $Z_0$ are linear combinations of elements in $C_0$ that are mapped to 0 via $\delta_0$. But, $\delta_0$ maps all the vertices to 0. Hence, all linear combinations of vertices will go to 0. Thus $Z_0 = C_0 = < v_1, v_2, v_3, v_4 >$.

- $B_0 = \text{image of } \delta_1 = < v_1 + v_2, v_2 + v_3, v_3 + v_4, v_1 + v_4, v_1 + v_3, v_2 + v_4 > = < v_1 + v_2, v_2 + v_3, v_3 + v_4 >$ since $v_1 + v_4, v_1 + v_3$ and $v_2 + v_4$ can be obtained from linear combinations of $v_1 + v_2, v_2 + v_3$, and $v_3 + v_4$.

- $H_0 = Z_0/B_0 = \{ v_1, v_2, v_3, v_4 \mid v_1 + v_2 = 0, v_2 + v_3 = 0, v_3 + v_4 = 0 \} = < [v_1] >$ where $[v_1] = \{ v_1, v_2, v_3, v_4 \}$ is a representative of the set containing all the vertices. Since we are working with coefficients in $\mathbb{Z}_2$, we get that $v_1 = v_2, v_2 = v_3, v_3 = v_4$.

  - Rank $C_0 = 4$
  - $|C_0| = 2^4 = 16$
  - $Z_0 = 4$
  - $|Z_0| = 2^4 = 16$
  - $B_0 = 3$
  - $|B_0| = 2^3 = 8$
  - $|H_0| = 2^1 = 2$

  - Find the matrix for $\delta_0$:
    $$M_0 = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
• Find the matrix for \( \delta_1 \):

\[
M_1 = \begin{pmatrix}
  v_1 & 1 & 0 & 0 & 1 & 0 & 1 \\
  v_2 & 1 & 1 & 1 & 0 & 0 & 0 \\
  v_3 & 0 & 1 & 0 & 1 & 1 & 0 \\
  v_4 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\]

• Simplify the matrix for \( \delta_1 \) so that the nonzero columns are linearly independent. Write the basis element above its corresponding column.

\[
M_1 = \begin{pmatrix}
  v_1 & 1 & 0 & 0 & 1 & 0 & 1 \\
  v_2 & 1 & 1 & 1 & 0 & 0 & 0 \\
  v_3 & 0 & 1 & 0 & 1 & 1 & 0 \\
  v_4 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix} \rightarrow \begin{pmatrix}
  v_1 & 1 & 0 & 0 & 0 & 0 & 1 \\
  v_2 & 1 & 1 & 1 & 0 & 0 & 0 \\
  v_3 & 0 & 1 & 0 & 0 & 0 & 0 \\
  v_4 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

• Find the matrix for \( \delta_2 \): 

\[
M_2 = \begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
  f_1 & f_2 & f_3 & f_4
\end{pmatrix}
\]

\[
M_2 = \begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
  f_1 & f_2 & f_3 & f_4
\end{pmatrix} \rightarrow \begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
  f_1 & f_2 & f_3 & f_4
\end{pmatrix}
\]

• Simplify the matrix for \( \delta_2 \) so that the nonzero columns are linearly independent. Write the basis element above its corresponding column.

\[
M_2 = \begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
  f_1 & f_2 & f_3 & f_4
\end{pmatrix} \rightarrow \begin{pmatrix}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
  f_1 & f_2 & f_3 & f_4
\end{pmatrix}
\]
• \( Z_1 = \{ \sum n_i e_i \text{ in } C_1 \mid \delta_1(\sum n_i e_i) = 0 \} = \langle e_1 + e_2 + e_4, e_2 + e_3 + e_5, e_1 + e_3 + e_6, e_4 + e_5 + e_6 \rangle \)

• Explain your answer for \( Z_1 \): 
  \( Z_1 \) = null space of \( M_1 = \langle e_1 + e_2 + e_4, e_2 + e_3 + e_5, e_1 + e_3 + e_6 \rangle \)

• \( B_1 = \) image of \( \delta_2 = \langle e_1 + e_2 + e_4, e_2 + e_3 + e_5, e_1 + e_3 + e_6 \rangle \)

• \( H_1 = Z_1 / B_1 = \langle e_1 + e_2 + e_4, e_2 + e_3 + e_5, e_1 + e_3 + e_6 \rangle = \langle 0 \rangle \)

• Rank \( C_1 = 6 \) \hspace{1cm} Rank \( Z_1 = 3 \) \hspace{1cm} Rank \( B_1 = 3 \)

• \( |C_1| = 2^6 = 64 \) \hspace{1cm} \( |Z_1| = 2^3 = 8 \) \hspace{1cm} \( |B_1| = 2^3 = 8 \)

• Rank \( H_1 = \text{Rank } Z_1 - \text{Rank } B_1 = 3 - 3 = 0 \)

• \( |H_1| = 2^0 = 1 \)

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• \( Z_2 = \{ \sum n_i f_i \text{ in } C_2 \mid \delta_2(\sum n_i f_i) = 0 \} = \langle f_1 + f_2 + f_3 + f_4 \rangle \)

• Explain your answer for \( Z_2 \):
  \( Z_2 \) = null space of \( M_2 = \langle f_1 + f_2 + f_3 + f_4 \rangle \)

• \( B_2 = \) image of \( \delta_3 = \langle 0 \rangle \) since \( \delta_3 : C_3 \to C_2 \) and \( C_3 = 0 \).

• \( H_2 = Z_2 / B_2 = \langle f_1 + f_2 + f_3 + f_4 \rangle = \langle f_1 + f_2 + f_3 + f_4 \rangle \)

• Rank \( C_2 = 4 \) \hspace{1cm} Rank \( Z_2 = 1 \) \hspace{1cm} Rank \( B_2 = 0 \)

• \( |C_2| = 2^4 = 16 \) \hspace{1cm} \( |Z_2| = 2^1 = 2 \) \hspace{1cm} \( |B_2| = 2^0 = 1 \)

• Rank \( H_2 = \text{Rank } Z_2 - \text{Rank } B_2 = 1 - 0 = 1 \)

• \( |H_2| = 2^1 = 2 \)
Figure 1: Barcode for $H_0$

Figure 2: Barcode for $H_1$