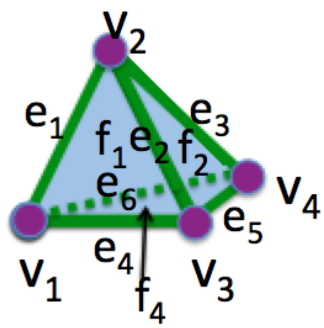
Homework 5: Solutions

Let C be the simplicial complex below (the boundary of a tetrahedron).



Find the following:

- $C_0 = \mathbb{Z}_2[v_1, v_2, v_3, v_4] = < v_1, v_2, v_3, v_4 >$
- $C_1 = \mathbb{Z}_2[e_1, e_2, e_3, e_4, e_5, e_6] = < e_1, e_2, e_3, e_4, e_5, e_6 >$
- $C_2 = \mathbb{Z}_2[f_1, f_2, f_3, f_4] = < f_1, f_2, f_3, f_4 >$
- $C_3 = < 0 >$

- $Z_0 = \{\Sigma_i n_i v_i \text{ in } C_0 \mid \delta_0(\Sigma_i n_i v_i) = 0\} = < v_1, v_2, v_3, v_4 > = C_0$
- Explain your answer for Z_0: The elements in Z_0 are linear combinations of elements in C_0 that are mapped to 0 via δ_0. But, δ_0 maps all the vertices to 0. Hence, all linear combinations of vertices will go to 0.
  Thus $Z_0 = C_0 =< v_1, v_2, v_3, v_4 >$.
- $B_0 =$ image of $\delta_1 = < v_1 + v_2, v_2 + v_3, v_3 + v_4, v_1 + v_4, v_1 + v_3, v_2 + v_4 > =$
  $< v_1 + v_2, v_2 + v_3, v_3 + v_4 >$ since $v_1 + v_4$, $v_1 + v_3$ and $v_2 + v_4$ can be obtained from linear combinations of $v_1 + v_2$, $v_2 + v_3$, and $v_3 + v_4$.
- $H_0 = Z_0/B_0 = \{v_1, v_2, v_3, v_4 \mid v_1 + v_2 = 0, v_2 + v_3 = 0, v_3 + v_4 = 0\} = < [v_1] >$ where $[v_1] = \{v_1, v_2, v_3, v_4\}$ is a representative of the set containing all the vertices. Since we are working with coefficients in $\mathbb{Z}_2$, we get that $v_1 = v_2$, $v_2 = v_3$, $v_3 = v_4$.
- Rank $C_0 = 4$ Rank $Z_0 = 4$ $B_0 = 3$
- $|C_0| = 2^4 = 16$ $|Z_0| = 2^4 = 16$ $|B_0| = 2^3 = 8$
- Rank $H_0 =$ Rank Z_0 - Rank $B_0 = 4 - 3 = 1$
- $|H_0| = 2^1 = 2$

- Find the matrix for δ_0:

$$M_0 = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Find the matrix for δ_1:

$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

- Simplify the matrix for δ_1 so that the nonzero columns are linearly independent. Write the basis element above its corresponding column.

$$M_1 = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array} \to \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_1+e_2+e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array} \to$$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_1+e_2+e_4 & e_2+e_3+e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \to$$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_1+e_2+e_4 & e_2+e_3+e_5 & e_1+e_3+e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- Find the matrix for δ_2 : $M_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$

- Simplify the matrix for δ_2 so that the nonzero columns are linearly independent. Write the basis element above its corresponding column.

$$M_2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \to \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_1+f_2+f_3+f_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

- $Z_1 = \{\Sigma_i n_i e_i \text{ in } C_1 \mid \delta_1(\Sigma_i n_i e_i) = 0\} = < e_1+e_2+e_4, e_2+e_3+e_5, e_1+e_3+e_6, e_4+e_5+e_6 >$
- Explain your answer for Z_1:
  $Z_1 =$ null space of $M_1 = < e_1 + e_2 + e_4, e_2 + e_3 + e_5, e_1 + e_3 + e_6 >$
- $B_1 =$ image of $\delta_2 =< e_1 + e_2 + e_4, e_2 + e_3 + e_5, e_1 + e_3 + e_6 >$
- $H_1 = Z_1/B_1 = \frac{<e_1+e_2+e_4, e_2+e_3+e_5, e_1+e_3+e_6>}{<e_1+e_2+e_4, e_2+e_3+e_5, e_1+e_3+e_6>} = < 0 >$
- Rank $C_1 = 6$ Rank $Z_1 = 3$ Rank $B_1 = 3$
- $|C_1| = 2^6 = 64$ $|Z_1| = 2^3 = 8$ $|B_1| = 2^3 = 8$
- Rank H_1 = Rank Z_1 - Rank $B_1 = 3 - 3 = 0$
- $|H_1| = 2^0 = 1$

- $Z_2 = \{\Sigma_i n_i f_i \text{ in } C_2 \mid \delta_2(\Sigma_i n_i f_i) = 0\} = < f_1 + f_2 + f_3 + f_4 >$
- Explain your answer for Z_2:

  $Z_2 =$ null space of $M_2 = < f_1 + f_2 + f_3 + f_4 >$
- $B_2 =$ image of $\delta_3 = < 0 >$ since $\delta_3 : C_3 \to C_2$ and $C_3 = 0$.
- $H_2 = Z_2/B_2 = \frac{<f_1+f_2+f_3+f_4>}{<0>} = < f_1 + f_2 + f_3 + f_4 >$
- Rank $C_2 = 4$ Rank $Z_2 = 1$ Rank $B_2 = 0$
- $|C_2| = 2^4 = 16$ $|Z_2| = 2^1 = 2$ $|B_2| = 2^0 = 1$
- Rank H_2 = Rank Z_2 - Rank $B_2 = 1 - 0 = 1$
- $|H_2| = 2^1 = 2$

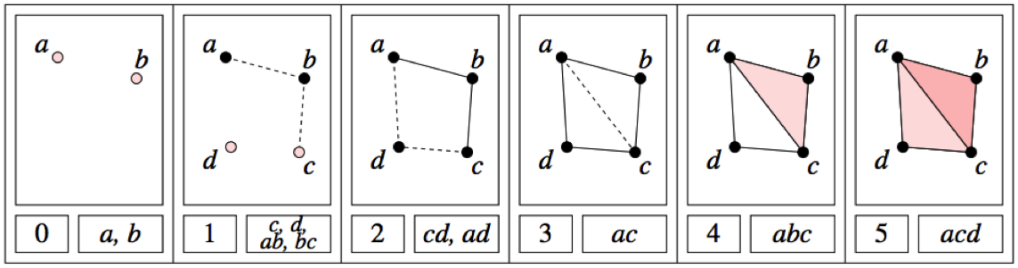


Figure 1: Barcode for H_0

Figure 2: Barcode for H_1