A quick example calculating the column space and the nullspace of a matrix.

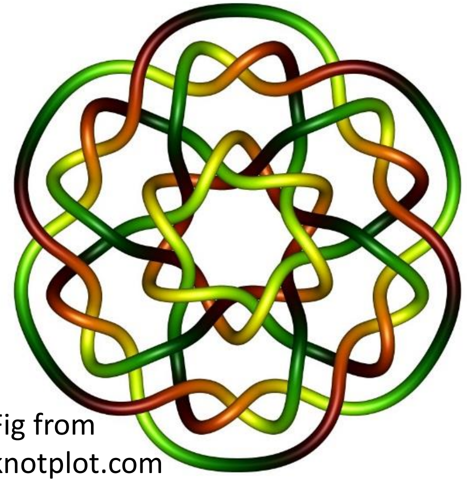

Fig from knotplot.com

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Determine the column space of A $= \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Column space of A

= span of the columns of A

= set of all linear combinations of the columns of A

Determine the column space of A $= \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Column space of A = col A =

col A = span $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix}, \begin{bmatrix} -42 \\ -32 \\ 87 \end{bmatrix} \right\}$

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$$= \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix} + c_3 \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} + c_4 \begin{bmatrix} -42 \\ -32 \\ 87 \end{bmatrix} \;\middle|\; c_i \text{ in } R \right\}$$

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Column space of A = col A =

$\text{col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ \ldots \end{bmatrix}, \begin{bmatrix} -10 \\ \ldots \end{bmatrix}, \ldots \right\}$

Want simpler answer

$= \left\{ c_1 \begin{bmatrix} \ldots \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix} + c_3 \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} + c_4 \begin{bmatrix} -42 \\ -32 \\ 87 \end{bmatrix} \,\middle|\, c_i \text{ in } R \right\}$

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Put A into echelon form:

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -10 & -24 & -42 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

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And determine the pivot columns

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A basis for col A consists of the 3 pivot columns from the **original** matrix A.

Thus basis for col A = $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix} \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$

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Thus col A is 3-dimensional.

Determine the column space of $A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

col A contains all linear combinations of the 3 basis vectors:

$$\text{col } A = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix} + c_3 \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \;\middle|\; c_i \text{ in } R \right\}$$

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Can you identify col A?

Determine the nullspace of $A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Put A into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -10 & -24 & -42 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow[\substack{R_1 - 2R_3 \to R_1 \\ R_3/3 \to R_3}]{R_1 + 8R_3 \to R_1} \begin{bmatrix} 1 & -10 & 0 & -18 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_2/2 \to R_2]{R_1 + 5R_2 \to R_1} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Nullspace of A = solution set of $A\mathbf{x} = \mathbf{0}$

Solve: $A\mathbf{x} = \mathbf{0}$ where $A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Put A into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -10 & -24 & -42 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

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Solve: $A\mathbf{x} = \mathbf{0}$ where $A \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_4 \\ -2x_4 \\ -x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix} x_4$$

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Thus Nullspace of A =

$$\text{Nul } A = \left\{ x_4 \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; x_4 \text{ in } \mathbf{R} \right\}$$

Solve: $A\mathbf{x} = \mathbf{0}$ where $A \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

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Thus Nullspace of A =

$$\text{Nul } A = \left\{ x_4 \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; x_4 \text{ in } \mathbf{R} \right\} = \text{span}\left\{ \begin{bmatrix} -2 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\}$$