7.2: Quadratic Forms $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is symmetric.

Example: $Q : R^2 \to R$

$$Q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x + 2y \\ 2x + 3y \end{bmatrix}$$

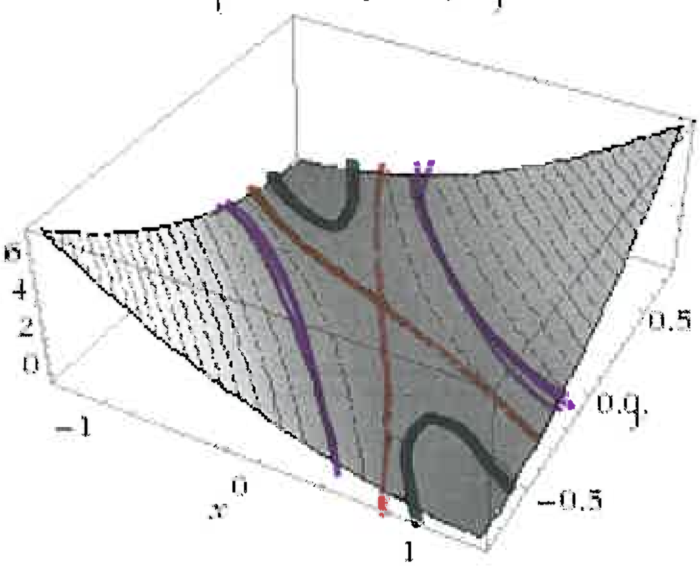
$$\boxed{Q(x,y) = x^2 + 4xy + 3y^2}$$

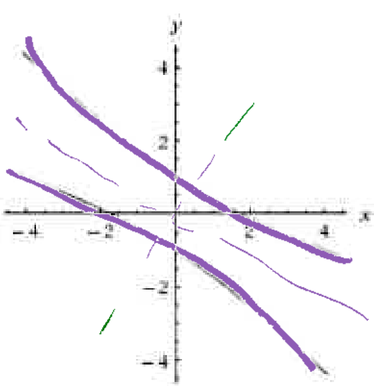
$$= x(x+2y) + y(2x+3y)$$

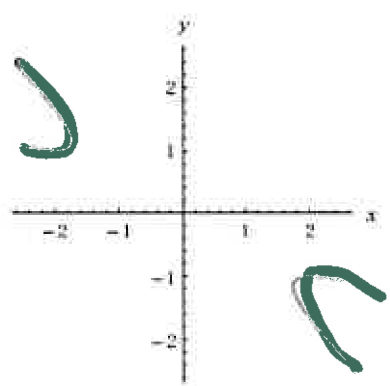
$$= x^2 + 2xy + 2xy + 3y^2$$

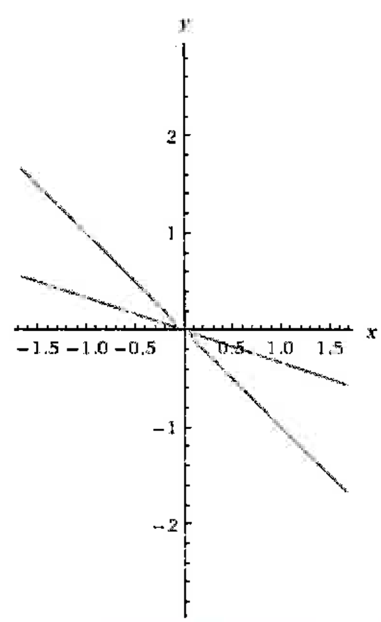
$$= 1x^2 + 4xy + 3y^2$$

$\{x^2 + 4xy + 3y^2\}$





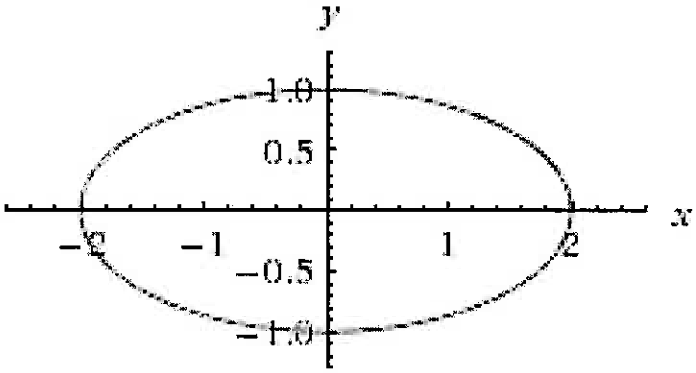


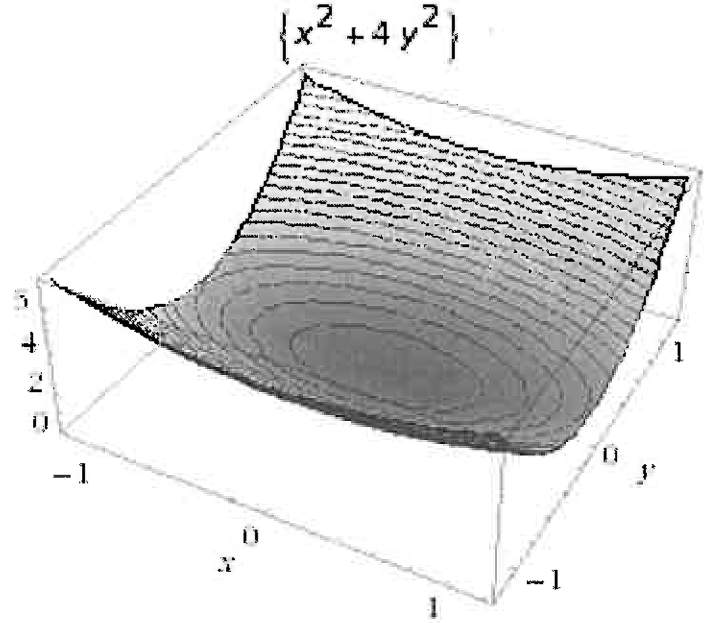


$\boxed{x^2 + 4xy + 3y^2 = 4}$ $x^2 + 4xy + 3y^2 = -1$ $\boxed{x^2 + 4xy + 3y^2 = 0}$

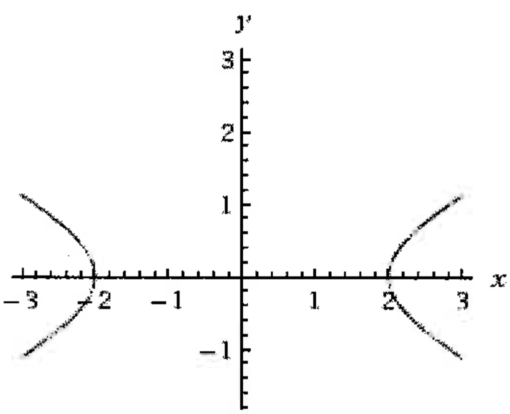
More examples: $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is symmetric.

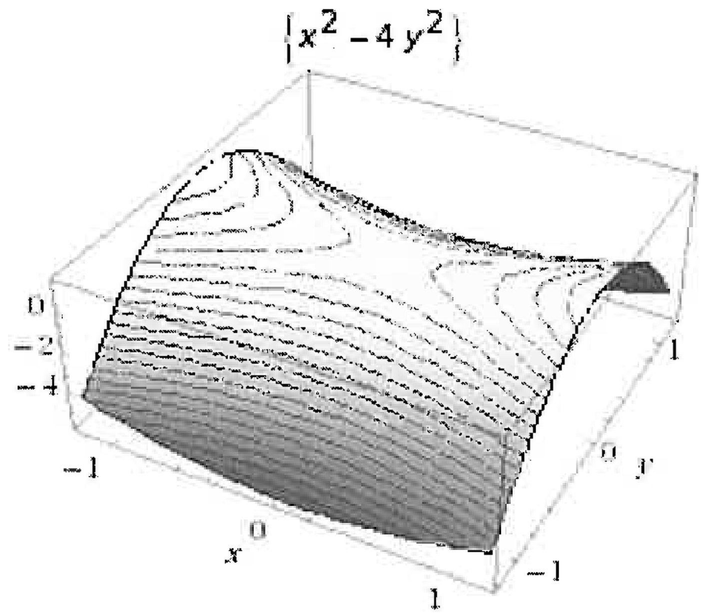
$$Q(x,y) = [\,x \quad y\,] \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

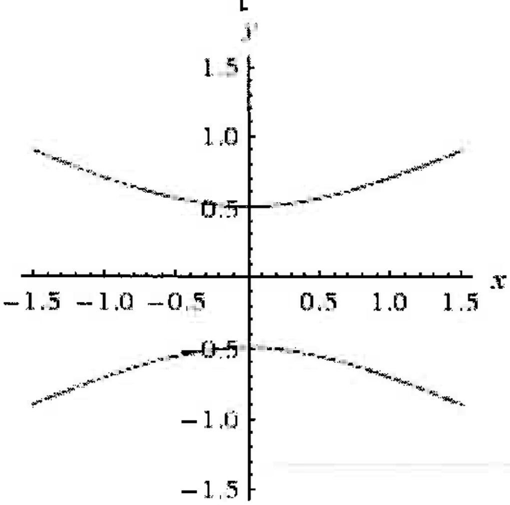


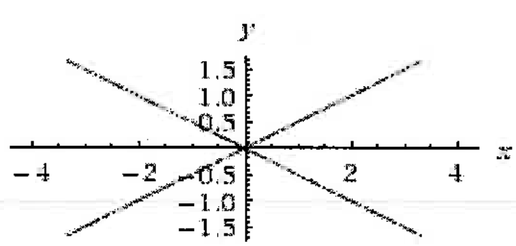


$$Q(x,y) = [\,x \quad y\,] \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

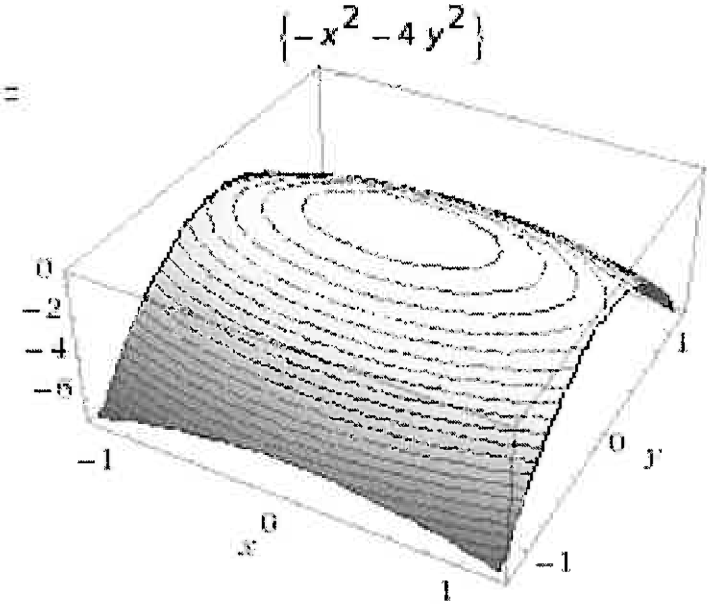


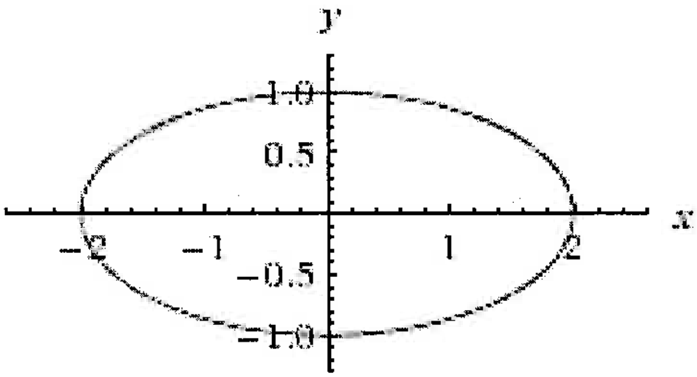




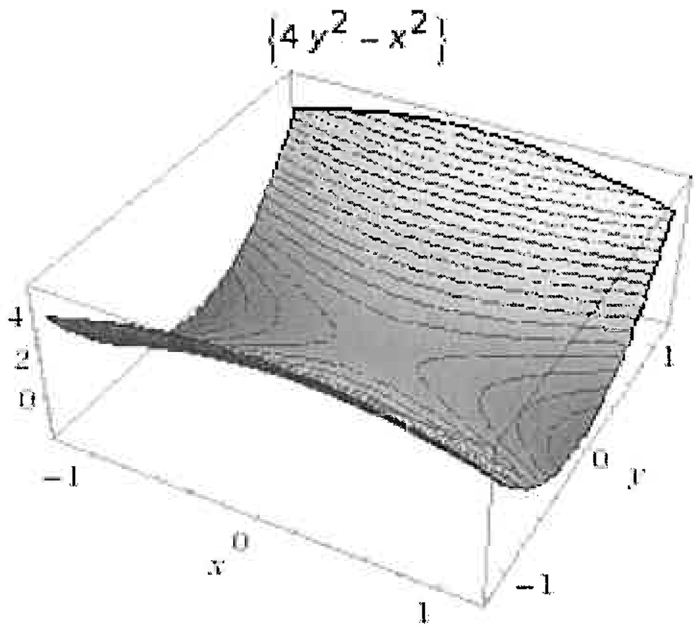


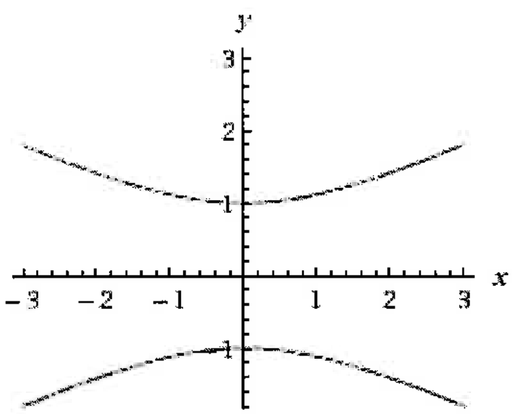
$$Q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$

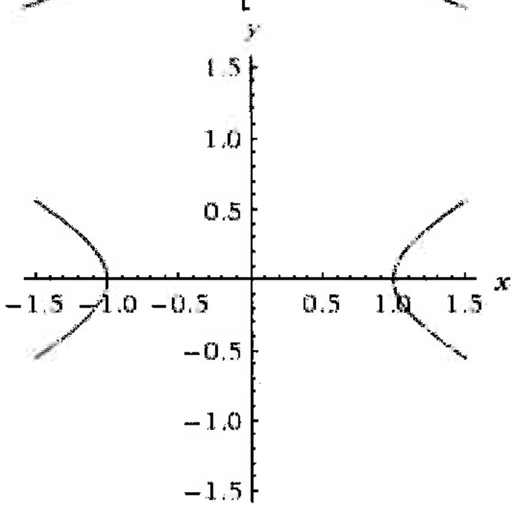


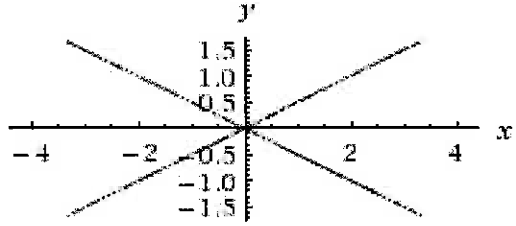


$$Q(x,y) = [\,x \quad y\,] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

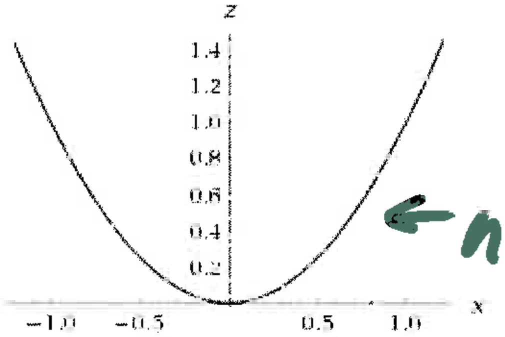






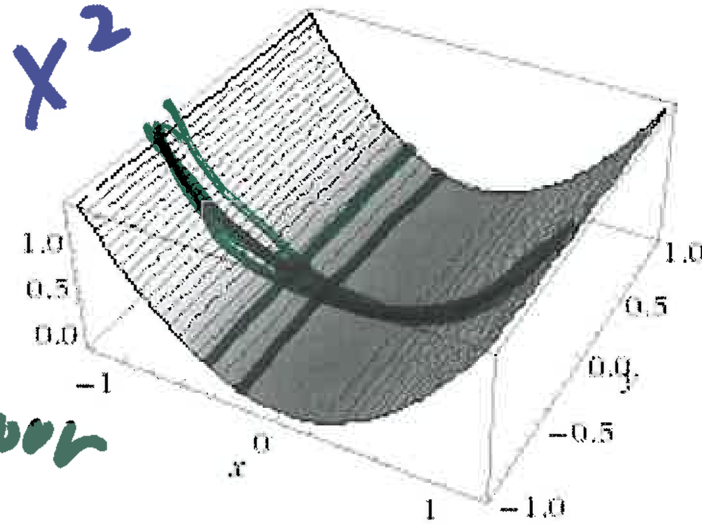


$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2$$

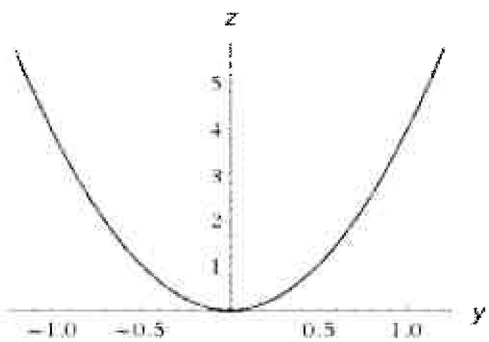


(x from −1 to 1)

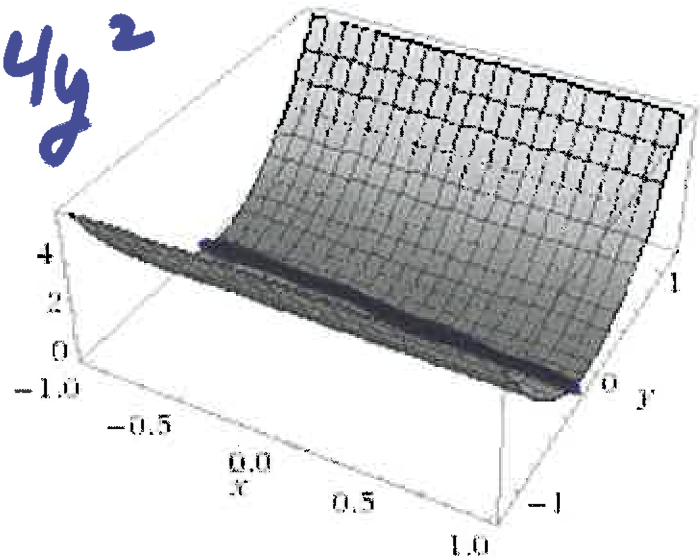
← not contour



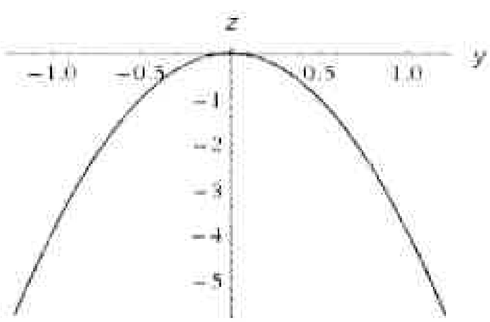
$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4y^2$$



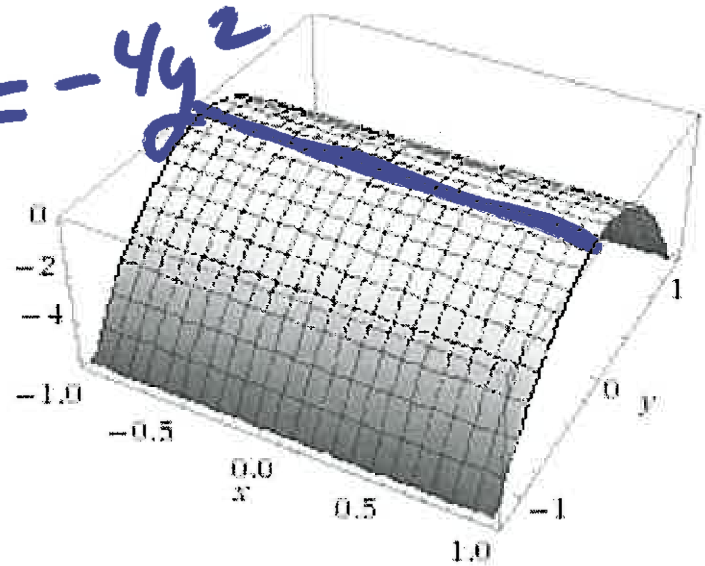
(y from −1 to 1)



$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -4y^2$$



(y from −1 to 1)



Defn and theorem:

A symmetric matrix A is **positive definite**

if and only if the $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$

if and only if all the eigenvalues of A are positive.

A symmetric matrix A is **negative definite**

if and only if the $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$

if and only if all the eigenvalues of A are negative.

A symmetric matrix A is **indefinite**

if and only if the $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values.

if and only if A are positive and negative eigenvalues.

A symmetric matrix A is **positive semidefinite**

if and only if the $\mathbf{x}^T A\mathbf{x} \geq 0$

≥ 0

if and only if all the eigenvalues of A are non-negative.

A symmetric matrix A is **negative semidefinite**

if and only if the $\mathbf{x}^T A\mathbf{x} \leq 0$ ≤ 0

if and only if all the eigenvalues of A are non-positive.

Change of variable:

Let $\mathbf{x} = P\mathbf{y}$.

$y^T D y$

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T AP\mathbf{y} = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T(P^T AP)\mathbf{y}$$

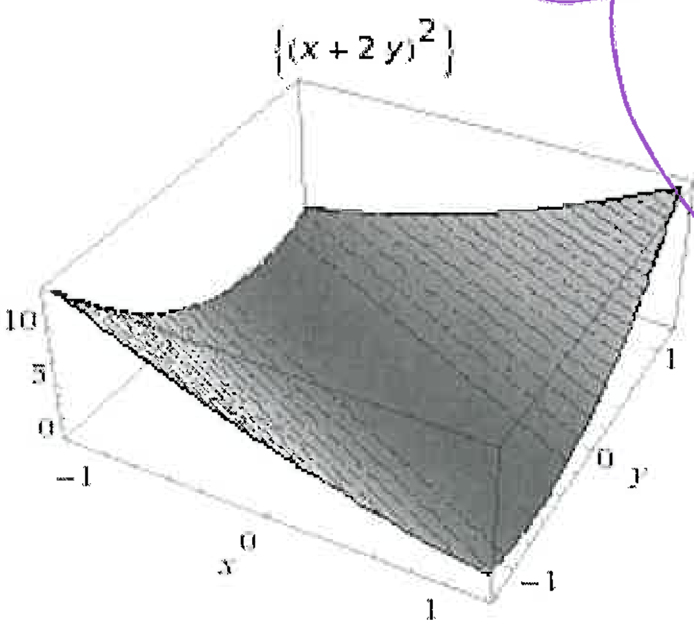
Suppose $A = PDP^{-1} = PDP^T$ where A is a symmetric matrix, D is diagonal, and P is orthonormal (i.e., $P^{-1} = P^T$).

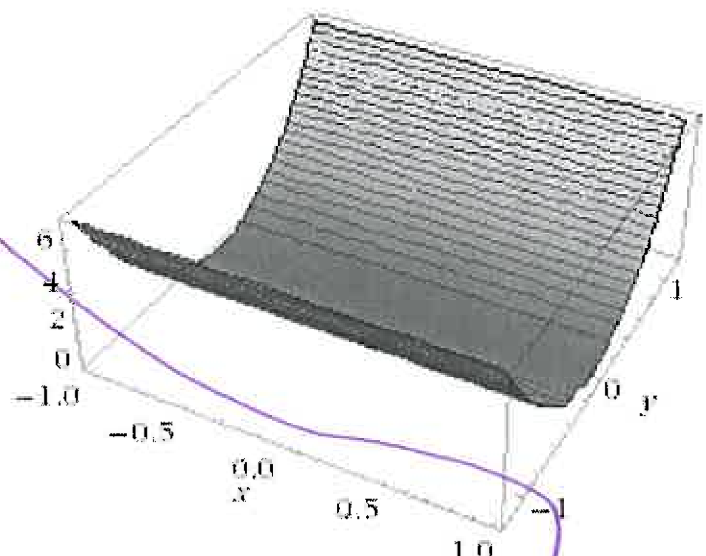
$A = PDP^T$ implies $P^T AP = P^T PDP^T P = D$

$$Q(\mathbf{y}) = \mathbf{y}^T(P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$$

$A \quad = \quad P \quad D \quad P^{-1}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$





$$Q(x, y) = [x \quad y] \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad Q(x, y) = [x \quad y] \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Make a change of variable that transforms the following quadratic form into a quadratic form with no cross-product term:

$$Q(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Step 1: ~~Orthogonally~~ normally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

See section 7.1:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = A = PDP^T = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Step 2: Let $\mathbf{x} = P\mathbf{y}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} y_1 + \frac{1}{\sqrt{5}} y_2 \\ \frac{1}{\sqrt{5}} y_1 + \frac{2}{\sqrt{5}} y_2 \end{bmatrix}$$

only rotating not stretching since unit vectors & orthogonal

After change of variable:

$$Q(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Example 1:

Orthogonally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

Thus $\lambda = 0, 5$ are are eigenvalues of A.

2.) Find a basis for each of the eigenspaces:

$$\lambda = 0 : (A - 0I) = A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 0.

$$\lambda = 0 : (A - 5I) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5.

3.) Create orthonormal basis:

Since A is symmetric and the eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ come from different eigenspaces (ie their eigenvalues are different), these eigenvectors are orthogonal. Thus we only need to normalize them:

$$\left|\left| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right|\right| = \sqrt{4+1} = \sqrt{5}$$

$$\left|\left| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right|\right| = \sqrt{1+4} = \sqrt{5}$$

Thus an orthonormal basis for $col(A) = R^2 = \left\{ \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$ [redacted]

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

Thus $P^{-1} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

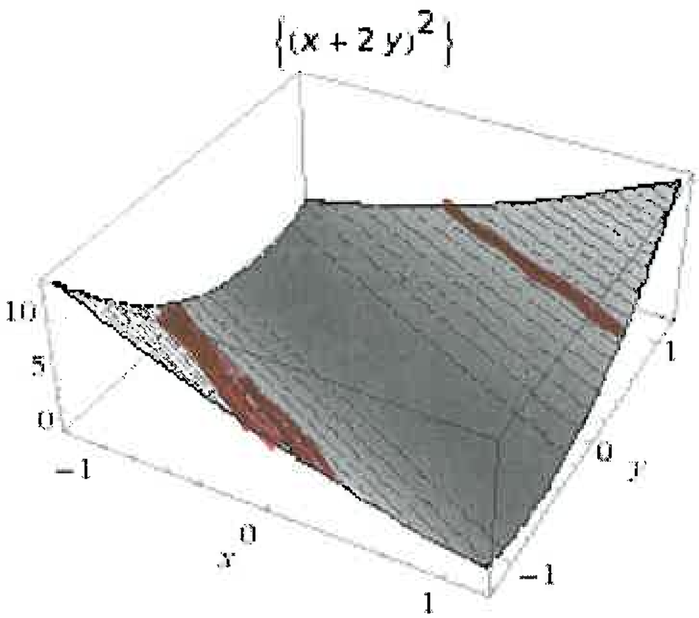
Note that in this example, $P^{-1} = P$, but that is NOT normally the case.

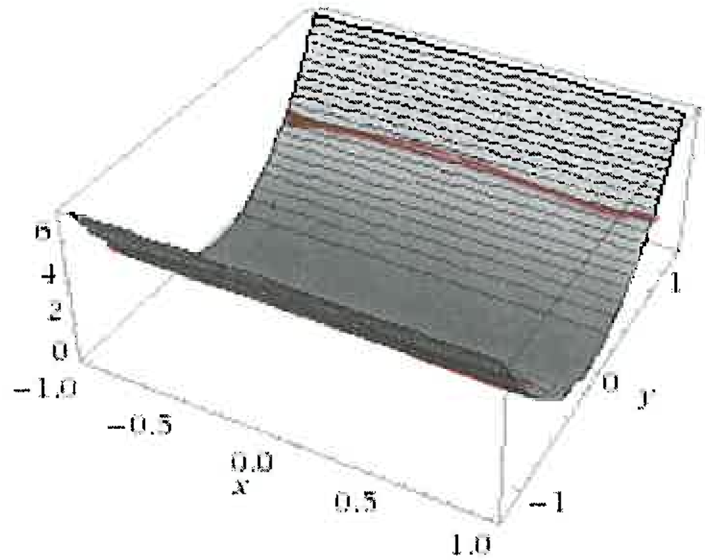
Thus $\boxed{A = PDP^{-1}}$

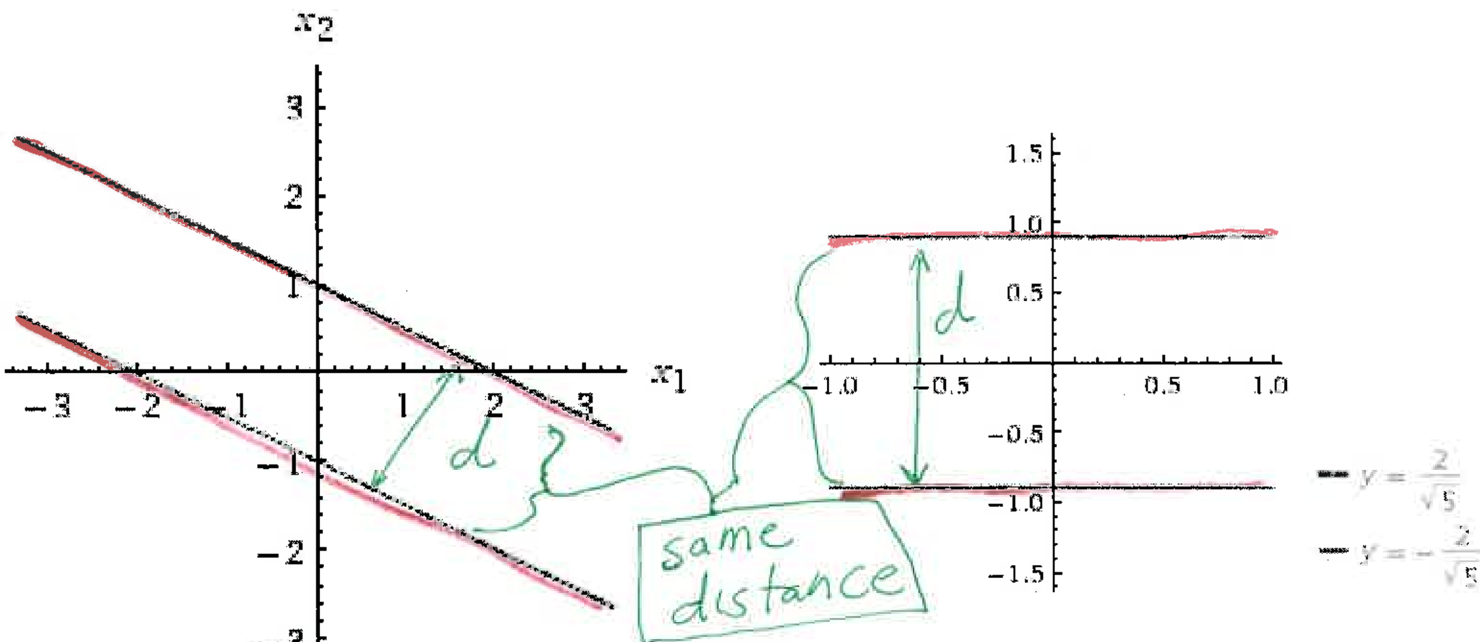
Thus $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

$=$

$$\begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$







$(x_1 + 2x_2)^2 = 4$

$((-\frac{2y_1}{\sqrt{5}} + \frac{y_2}{\sqrt{5}}) + 2(\frac{y_1}{\sqrt{5}} + \frac{2y_2}{\sqrt{5}}))^2 = 4$