5.3: Diagonalization

$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \quad A^2 \neq \begin{pmatrix} 1^2 & 2^2 \\ (-1)^2 & 1^2 \end{pmatrix}$

Note that multiplying diagonal matrices is easy:

Let $D = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$. Then

$\begin{bmatrix} v_1 & \boxed{v_2} \end{bmatrix} \begin{bmatrix} x_1 \\ \boxed{x_2} \end{bmatrix} = x_1 v_1 + x_2 v_2$

$$D^2 = \begin{bmatrix} 10 & \boxed{0} \\ 0 & \boxed{-1} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ \boxed{0} & \boxed{-1} \end{bmatrix} = \begin{bmatrix} 10\begin{pmatrix} 10 \\ 0 \end{pmatrix} + \cancel{0\begin{pmatrix} 0 \\ -1 \end{pmatrix}} & \cancel{0\begin{pmatrix} 10 \\ 0 \end{pmatrix}} + (-1)\begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 10^2 & 0 \\ 0 & (-1)^2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} 10^k & 0 \\ 0 & (-1)^k \end{bmatrix}$$

Defn: The matrices A and B are **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$. ■

Defn: A matrix A is **diagonalizable** if A is similar to a diagonal matrix. $A \longrightarrow P \quad P^{-1}AP = D$

I.e. A is diagonalizable if there exists an invertible matrix P such that $\underline{P^{-1}AP = D}$ where D is a diagonal matrix.

Application: Calculating A^k.

$$\underbrace{(P P^{-1}}_{I} A \underbrace{P P^{-1})}_{I} = PDP^{-1} \Rightarrow A$$

$k = 1:\ A = \cancel{P^{-1}AP} \quad PDP^{-1}.$

$k = 2:\ A^2 = PD\underbrace{P^{-1}P}_{I}DP^{-1} = PDDP^{-1} = PD^2P^{-1}.$

$k = 3:\ A^3 = (PD\underbrace{P^{-1})(P}DP\underbrace{^{-1})(P}DP^{-1}) = PD^3P^{-1}.$

Similarly $A^k = PD^kP^{-1}$

Example:

Let $D = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix}$

Then $\begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$

$A \quad = \quad P \quad D \quad P^{-1}$

Thus, $A^3 = P D^3 P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 10^3 & 0 \\ 0 & (-1)^3 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 \\ -1000 & -5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \left[\begin{pmatrix} 0 \\ -5000 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1000 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ -5 \end{pmatrix} \right]$$

$$= \begin{bmatrix} -1 & 0 \\ -5005 & 1000 \end{bmatrix}$$

Equivalent Questions:

- Given an $n \times n$ matrix, does there exist a basis for R^n consisting of eigenvectors of A?
- Given an $n \times n$ matrix, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?

Thm: Let A be an $n \times n$ matrix. The following are equivalent:

a.) A is diagonalizable.

b.) A has n linearly independent eigenvectors.

c.) There exists a basis for R^n consisting of eigenvectors of A.

$\Leftrightarrow A = PDP^{-1}$

Sec 5.1 & 5.2

$Ax = \lambda x$

Example: Suppose $AP = PD$ where

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\text{Then } PD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} = \left(5\begin{pmatrix} 1 \\ 3 \end{pmatrix} \;\; 6\begin{pmatrix} 2 \\ 4 \end{pmatrix} \right)$$

$$\text{Thus } AP = A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \left[A\begin{pmatrix} 1 \\ 3 \end{pmatrix} \;\; A\begin{pmatrix} 2 \\ 4 \end{pmatrix} \right]$$

$$\text{Hence } A \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and } A \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Thus an eigenvalue of A = 5 with eigenvector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Another eigenvalue of A = 6 with eigenvector $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

Thus if $AP = PD$, then
if the diagonal entries of D are $d_1, ..., d_n$

and the i^{th} column of P is an eigenvector

corresponding to the eigenvalue d_i.

Note P is an invertible SQUARE matrix where columns P are eigenvector of the matrix A

To diagonalize a matrix A:

1.) Find the eigenvalues of A.

$$\text{Solve } det(\lambda I - A) = 0 \text{ for } \lambda.$$

2.) Find a basis for each of the eigenspaces.

$$\text{Solve } (\lambda_j I - A)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Case 3a.) IF the geometric multiplicity is LESS then the algebraic multiplicity for at least ONE eigenvalue of A, then A is NOT diagonalizable. (Cannot find square matrix P).

Case 3b.) IF the geometric multiplicity equals the algebraic multiplicity for ALL the eigenvalues of A, then A is diagonalizable. Thus,

- Use the eigenvalues of A to construct the diagonal matrix D
- Use the basis of the corresponding eigenspaces for the corresponding columns of P. (NOTE: P is a SQUARE matrix).

NOTE: ORDER MATTERS.

Examples:

$$A = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thm: Suppose λ_i, $i = 1, ..., n$ are DISTINCT eigenvalues of a matrix A. If $\mathcal{B}_i$ is a basis for the eigenspace corresponding to λ_i, then

$\mathcal{B} = \mathcal{B}_1 \cup ... \cup \mathcal{B}_n$ is linearly independent.

Defn: Suppose the characteristic polynomial of A is

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2}...(\lambda - \lambda_n)^{k_n}$$

where the λ_i, $i = 1, ..., n$ are DISTINCT. Then the **algebraic multiplicity of** λ_i is k_i.

That is the **algebraic multiplicity of** λ_i is the number of times that $(\lambda - \lambda_i)$ appears as a factor of the characteristic polynomial of A.

Defn: The **geometric multiplicity of** λ_i = dimension of the eigenspace corresponding to λ_i.

Thm (Geometric and Algebraic Multiplicity):

a.) The geometric multiplicity is less than or equal to the algebraic multiplicity [That is, Nullity of $(\lambda_i I - A) \leq k_i$].

b.) A is diagonalizable if and only if the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue.