Basis for Nul E = $\left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\5\\-7\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-5\\3\\0\\0\\1\\1\end{bmatrix} \right\}$

Solve: E x = 0 where E ~ $\begin{bmatrix} 0 & 1 & 0 & -5 & 0 & 0 & 5 \\ 0 & 0 & 1 & 7 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

REF →

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix} = \begin{bmatrix}x_1\\5x_4-5x_7\\-7x_4+3x_7\\x_4\\0\\x_7\\x_7\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\end{bmatrix}x_1 + \begin{bmatrix}0\\5\\-7\\1\\0\\0\\0\end{bmatrix}x_4 + \begin{bmatrix}0\\-5\\3\\0\\0\\1\\1\end{bmatrix}x_7$$

Nul E = span $\left\{ \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\5\\-7\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-5\\3\\0\\0\\1\\1\end{bmatrix} \right\}$ is L.I.

If ~~S~~ solution set written only in terms of free variables, than get lin indep for free. Ie

Determine the column space of E where E ~ $\begin{bmatrix} 0 & 1 & 0 & -5 & 0 & 0 & 5 \\ 0 & 0 & 1 & 7 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

IF POSSIBLE

Note: We don't know the original matrix E. We only know REF of E.

NOT POSSIBLE

col E is 4-dim subspace of $\mathbb{R}^5$

5 x 7

Determine the column space of B where B ~ $\begin{bmatrix} 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

3x5

IF POSSIBLE

Note: We don't know the original matrix B. We only know REF of B.

But col B is a 3-dim subspace of $\mathbb{R}^3$

$\Rightarrow$ col B = $\mathbb{R}^3$

Basis for col B = $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

or $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}, \begin{bmatrix}5\\6\\7\end{bmatrix} \right\}$

Determine the column space of A = $\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Column space of A = col A =

col A = span $\left\{ \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}-10\\-8\\20\end{bmatrix}, \begin{bmatrix}-24\\-18\\51\end{bmatrix}, \begin{bmatrix}-42\\-32\\87\end{bmatrix} \right\}$

SIMPLIFY

$= \left\{ c_1\begin{bmatrix}1\\1\\-2\end{bmatrix} + c_2\begin{bmatrix}-10\\-8\\20\end{bmatrix} + c_3\begin{bmatrix}-24\\-18\\51\end{bmatrix} + c_4\begin{bmatrix}-42\\-32\\87\end{bmatrix} \;\middle|\; c_i \text{ in } \mathbb{R} \right\}$

Determine the column space of A = $\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Put A into echelon form:

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -10 & -24 & -42 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

A basis for col A consists of the 3 pivot columns from the **original** matrix A.

Thus basis for col A = $\left\{ \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}-10\\-8\\20\end{bmatrix}, \begin{bmatrix}-24\\-18\\51\end{bmatrix} \right\}$

other Basis for colA

col A = $\mathbb{R}^3$

λ e. value
$\vec{x}$ e vector if $\vec{x} \neq 0$ is a sol'n to

$A\vec{x} = \lambda \vec{x}$ $\qquad$ $A\vec{x} = \lambda I \vec{x}$

Thus to find the eigenvalues of A and their corresponding eigenvectors:

$(A - \lambda I)\vec{x} = 0$

Step 1: Find eigenvalues: Solve the equation

$$\boxed{det(A - \lambda I) = 0 \text{ for } \lambda.}$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(A - \lambda_0 I)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(A - \lambda I) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(A - \lambda_0 I)\mathbf{x} = \mathbf{0}$.

Eigenspace for λ_0 = Nulspace of $(A - \lambda_0 I)$

Note: An eigenspace is a vector space

The vector $\vec{\mathbf{0}}$ is always in the eigenspace.

The vector $\vec{\mathbf{0}}$ is never an eigenvector.

The number 0 can be an eigenvalue. $\quad det(A - 0I) = 0$

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

$\Rightarrow Ax = 0x$ has a non zero sol'n

Thm: If $A\mathbf{x} = \lambda\mathbf{x}$, then $A^k\mathbf{x} = \lambda^k\mathbf{x}$. That is, if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$, then λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$ where k is any integer.

SKIP

Defn: Suppose the characteristic polynomial of A is

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2}...(\lambda - \lambda_n)^{k_p} = det(A - \lambda I)$$

where the λ_i, $i = 1, ..., p$ are DISTINCT. Then the **algebraic multiplicity of** λ_i is k_i.

That is the **algebraic multiplicity of** λ_i is the number of times that $(\lambda - \lambda_i)$ appears as a factor of the characteristic polynomial of A.

Defn: The **geometric multiplicity of** λ_i
= dimension of the eigenspace corresponding to λ_i.

Thm (Geometric and Algebraic Multiplicity): The geometric multiplicity is less than or equal to the algebraic multiplicity [That is, Nullity of $(A - \lambda_i I) \leq k_i$].

Find e. valu & e. vectors fo

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$= \begin{vmatrix} 1-\lambda & 2 & 3 & 4 \\ 0 & 2-\lambda & 6 & 5 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)^{3} = 0$$

$$\lambda = 1, 2$$

repeated e. value w/ multiplicity 3

Find e. vectors con to e. value $\lambda = 1$

Solve $(A-\lambda I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cccc|c} 0 & 2 & 3 & 4 & 0 \\ 0 & 1 & 6 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 - 4R_4$

$R_2 - 5R_4$

$R_1 - 3R_3$

$R_1 - 6R_3$

$$\sim \left[\begin{array}{cccc|c} 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

f.v.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus for e. value $\lambda = 1$,

its eigenspace = span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Some examples of e. vectors w/ e. value $\lambda = 1$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \pi \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Find e. space corresponding to e. value $\lambda = 2$

I.e. solve $(A - 2I)\vec{x} = 0$

$$\begin{array}{ccccc} & & 3 & -5/2 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} -1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 6 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} +1 & -2 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 5/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccccc} & \text{f.V.} & & \text{f.V.} & \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + \frac{3}{2}x_4 \\ x_2 \\ (-5/6)x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 3/2 \\ 0 \\ -5/6 \\ 1 \end{bmatrix} x_4$$

Thus for e. value $\lambda = 2$:

$$\text{Eigen space} = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 0 \\ -5/6 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \\ -5 \\ 6 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ 1 \\ -5 \\ 6 \end{bmatrix} \right\}$$

Note many answers are correct. Just NEED (1) span (2) lin indep

The above three examples have the same span & are l. indep

Examples of e. vectors w/ e. value $\lambda = 2$

1) $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ check: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}$

2) $\begin{bmatrix} 9 \\ 0 \\ -5 \\ 6 \end{bmatrix}$ check: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 9 \\ 0 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 9-15+24 \\ -30+30 \\ -10 \\ 12 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ -10 \\ 12 \end{bmatrix}$

3) $\begin{bmatrix} 11 \\ 1 \\ -5 \\ 6 \end{bmatrix}$ Check: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 6 & 5 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 1 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 11+2-15+24 \\ 2-30+30 \\ -10 \\ 12 \end{bmatrix} = \begin{bmatrix} 22 \\ 2 \\ -10 \\ 12 \end{bmatrix}$

NOTE ANY VECTOR IN THE EIGENSPACE EXCEPT THE ZERO VECTOR is AN EIGENVECTOR.