Ch 2 partial review:

Recall W is a **subspace** of R^n (**vector space**) if W is closed under scalar multiplication and vector addition.

closed under linear combinatn

I.e., W is a subspace of R^n if

$$\mathbf{v_1}, \mathbf{v_2} \text{ in } W \text{ implies } c_1\mathbf{v_1} + c_2\mathbf{v_2} \text{ in } W.$$

Note if W is a finite dimensional subspace, ~~then~~ iff for some vectors $\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}$ in W:

$$\begin{aligned} W &= span\{\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}\} \\ &= \{c_1\mathbf{w_1} + c_2\mathbf{w_2} + ... + c_k\mathbf{w_k} \mid c_i \in R\} \\ &= \text{the set of all linear combinations of the vectors} \\ &\qquad \mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}. \end{aligned}$$

Examples:

$= span\{a_1, ... a_n\}$

The column space of $A = [\mathbf{a_1}\ \mathbf{a_2}\ ...\ \mathbf{a_n}]$

$$= \{c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} \mid c_i \in R\}$$

$$= \{\mathbf{b} \mid A\mathbf{x} = \mathbf{b} \text{ has at least one solution }\}$$

is a subspace.

$$[\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{b} \text{ in col } A$$

Nullspace of A = solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace:

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{0}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is also a solution:

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2)$$

$$= c_1 \underbrace{A\vec{v}_1}_{0} + c_2 \underbrace{A\vec{v}_2}_{0} = 0$$

The solution set of $A\mathbf{x} = \mathbf{b}$ is NOT a subspace unless $\mathbf{b} = \mathbf{0}$:

NOTE $\vec{0}$ is not is soln set so soln set to $Ax=b \neq 0$ is NOT A SUBSPACE

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is a solution to

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 \underbrace{A\vec{v}_1} + c_2 \underbrace{A\vec{v}_2}$$

$$= c_1\vec{b} + c_2\vec{b} = (c_1+c_2)\vec{b} \neq \vec{b}$$

unless $\vec{b} = 0$ or $c_1+c_2=1$

The soln set to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2+1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a soln to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so $3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a soln

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Ch 5: The eigenspace corresponding to an eigenvalue λ is a subspace.

Thus soln set to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is NOT a subspace

Claim the sol'n set to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

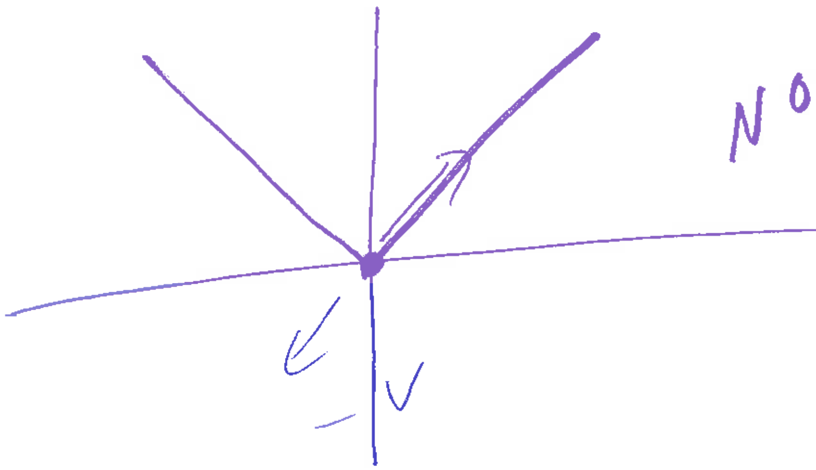
NOT A SUBSPACE

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the sol'n set since $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

But $0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is not in the sol'n set since

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus soln set to $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is <u>NOT</u> a subspace



NOT A SUBSPACE

Defn: Let W be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for W if

i.) $\mathcal{T}$ is linearly independent and ← small enough

ii.) $\mathcal{T}$ spans W. ← large enough

goldilocks

I.e.,
$\mathcal{T}$ is the smallest collections of vectors that span W.

Basis thm: Let W be a p-dimensional subspace of R^n.

i.) If $W = \text{span}\{w_1, ..., w_p\}$, then $\{w_1, ..., w_p\}$ is a basis for W.

ii.) If $v_1, ..., v_p$ are linearly independent vectors in W, then $\{v_1, ..., v_p\}$ is a basis for W.

Thm: All basis for a finite-dimensional vector space have the same number of elements.

Defn:
$dim(V)$ = the **dimension** of a finite-dim vector sp V
= the number of vectors in any basis for V.

If $dim(V) = n$, then V is said to be n-dimensional.

rank A = Rank of a matrix A = dimension of Col A
= number of pivot columns of A.

nullity of A = dimension of Nul A
= number of free variables.

$Ax = 0$

Rank(A) + nullity(A) = Number of columns of A.

That is,

The number of pivots of A	+	The number of free variables of A	=	The number of columns of A

Ex. 1) Suppose A is a $5 X 7$ matrix.

If Rank(A) = 4, then nullity(A) = $7 - 4 = \boxed{3}$

$A\mathbf{x} = \mathbf{0}$ has infinite # of solutions.

$A\mathbf{x} = \mathbf{b}$ has no sol'n or ∞ # of solutions.

If Rank(A) = 5, then nullity(A) = $7 - 5 = \boxed{2}$

$A\mathbf{x} = \mathbf{0}$ has ∞ # of solutions.

$A\mathbf{x} = \mathbf{b}$ has ∞ # of solutions.

If Rank(A) = 5, the column space of A = $\mathbb{R}^5$

col A is a 5-dim subspace of $\mathbb{R}^5$.

Thm 8′: If A is a $\boxed{\text{SQUARE}}$ $n \times n$ matrix, then the following are equivalent.

a.) A is invertible.

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries
[I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row).

d.) The column vectors of A are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any b.

g.) $Ax = b$ has a unique sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n.
[every vector in R^n can be written as a linear combination of the columns of A].

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

m.) A^T is invertible.

n.) A is expressible as a product of elementary matrices.

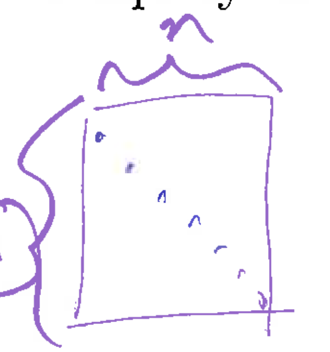
pivot in every row

o.) The column vectors of A form a basis for R^n. [every vector in R^n can be written uniquely as a linear combination of the columns of A].

p.) Col $A = R^n$.

q.) dim Col $A = n$.

r.) rank of $A = n$.



pivot in every column

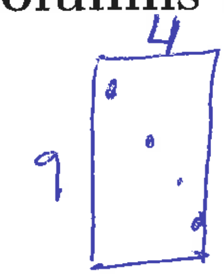
s.) Nul $A = \{\mathbf{0}\}$, ✓

t.) dim Nul $A = 0$. ✓

u.) A has nullity 0. ✓

no free variables

Rank(A) + nullity(A) = Number of columns of A.



Ex. 2) Suppose A is a $9X4$ matrix.

If Rank(A) $= 4$, then nullity(A) $= 4 - 4 = 0$

$A\mathbf{x} = \mathbf{0}$ has ___unique___ solution~~s~~.

$A\mathbf{x} = \mathbf{b}$ has ___at most one___ solution~~s~~.

none or one

If Rank(A) $= 3$, then nullity(A) $= 4 - 3 = 1$

$A\mathbf{x} = \mathbf{0}$ has ___∞ # of___ solutions.

$A\mathbf{x} = \mathbf{b}$ has ___no soln or ∞ # of___ solutions.

A is a 9×4 matrix

RANK(A) = 4

row equiv

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

column space is a 4-dim subspace of $\mathbb{R}^9$

Nulspace is a $\underline{0}$-dim subspace $\mathbb{R}^4$

$= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3.3: Cramer's Rule, Adjoint, Inverses, Area

Defn: Let $A_i(\mathbf{b})$ = the matrix derived from A by replacing the i^{th} column of A with $\mathbf{b}$.

Cramer's Rule: Suppose $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ matrix such that $det A \neq 0$. Then

$$\boxed{x_i = \frac{det A_i(\mathbf{b})}{det A}.}$$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4) - (3)(2) = 4 - 6 = -2$$

$$det \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix} = (5)(4) - (6)(2) = 20 - 12 = 8$$

$$det \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix} = (1)(6) - (3)(5) = 6 - 15 = -9$$

Thus $x_1 = \frac{8}{-2} = -4, \quad x_2 = \frac{-9}{-2} = \frac{9}{2}.$

Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$$\begin{vmatrix} 5 & 2 \\ 6 & 4 \end{vmatrix} = 20 - 12 = 8$$

$$\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = 6 - 15 = -9$$

$$x_1 = \frac{8}{-2} = -4$$

$$x_2 = \frac{-9}{-2} = 9/2$$

Observe for 2×2 case: If $A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{matrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{matrix}}_{Ax=b} & \begin{matrix} a_{12} \\ a_{22} \end{matrix} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} x_1 & 0 \\ x_2 & 1 \end{bmatrix}$$

$$AI_1(\mathbf{x}) = A_1(\mathbf{b})$$

$$det(AI_1(\mathbf{x})) = det(A_1(\mathbf{b}))$$

$$det(A)\ det(I_1(\mathbf{x})) = det(A_1(\mathbf{b}))$$

$$det(A)\ x_1 = det(A_1(\mathbf{b}))$$

Let $\vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ 1 in ith spot

Thus $x_1 = \frac{det(A_1(\mathbf{b}))}{det(A)}$

$I_j(X) = [\vec{e}_1\ \vec{e}_2\ X \ldots \vec{e}_n]$

$$AI_j(\mathbf{x}) = [A\mathbf{e_1}\ \ldots\ A\mathbf{e_{j-1}}\ A\mathbf{x}\ A\mathbf{e_{j+1}}\ \ldots\ A\mathbf{e_n}] = A_j(\mathbf{b})$$

$[\vec{a}_1 \ldots \vec{a}_{j-1}\ \vec{b}\ \vec{a}_{j+1} \ldots \vec{a}_n] = A_j(b)$

Solve the following using Cramer's rule:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 4 & 10 & 0 & 4 & 10 \\ 5 & 0 & 6 & 5 & 0 \end{array}\right| = 60 + 0 + 0 - 150 - 0 - 48$$

$$= 60 - 198 = -138$$

$$\begin{vmatrix} 0 & 2 & 3 \\ 0 & 10 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

$$\begin{vmatrix} & 0 & \\ & 0 & \\ & 0 & \end{vmatrix}$$

$$\begin{vmatrix} & & 0 \\ & & 0 \\ & & 0 \end{vmatrix}$$