Suppose $A\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} = \mathbf{0}$, then $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$

NOTE: Nullspace of A = span$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.8 Subspaces of R^n.

Long definition emphasizing important points:

Defn: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if and only if the following three conditions are satisfied:

i.) $\mathbf{0}$ is in W,

ii.) if $\mathbf{v_1}, \mathbf{v_2}$ in W, then $\mathbf{v_1} + \mathbf{v_2}$ in W,

iii.) if $\mathbf{v}$ in W, then $c\mathbf{v}$ in W for any scalar c.

Short definition: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if $\mathbf{v_1}, \mathbf{v_2}$ in W implies $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ in W,

---

Note that if S is a subspace, then

if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ in S, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$ is in S.

$0\mathbf{v} = \mathbf{0}$ is in S.

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if
i.) $\mathcal{T}$ is linearly independent and
ii.) $\mathcal{T}$ spans S.

Subspace of $\mathbb{R}^n$ = Vector Space

Defn S is a subspace (vector space)

if v_1, v_2 in S

$\Rightarrow c v_1 + c v_2$ in S

$\cancel{\Rightarrow 0}$

NOTE S is a subspace

$\Rightarrow 0 \in S$

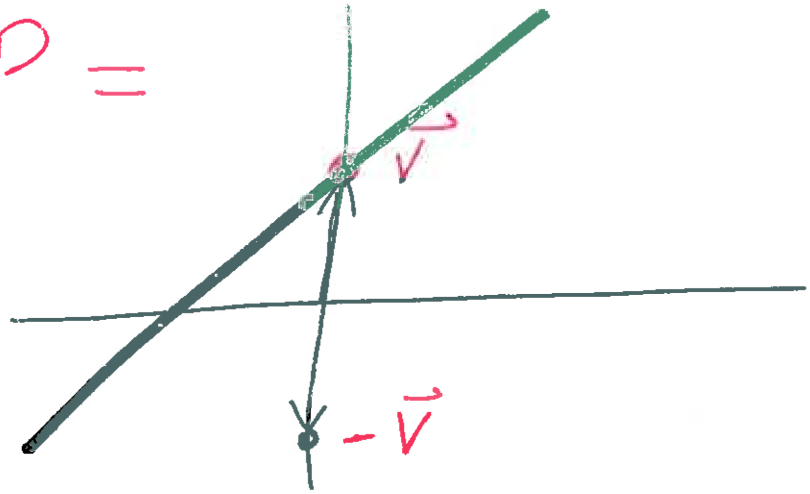
$$0 \vec{v} = \vec{0}$$

real #

scalar

vector

Ex

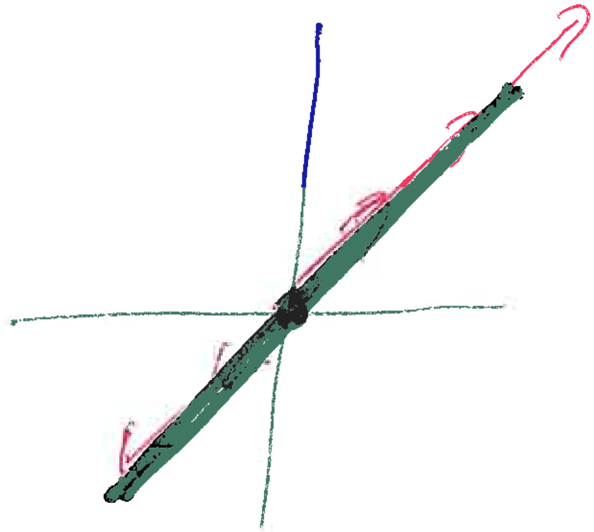
$P =$ 

NOT A SUBSPACE

0 is not on line ✓

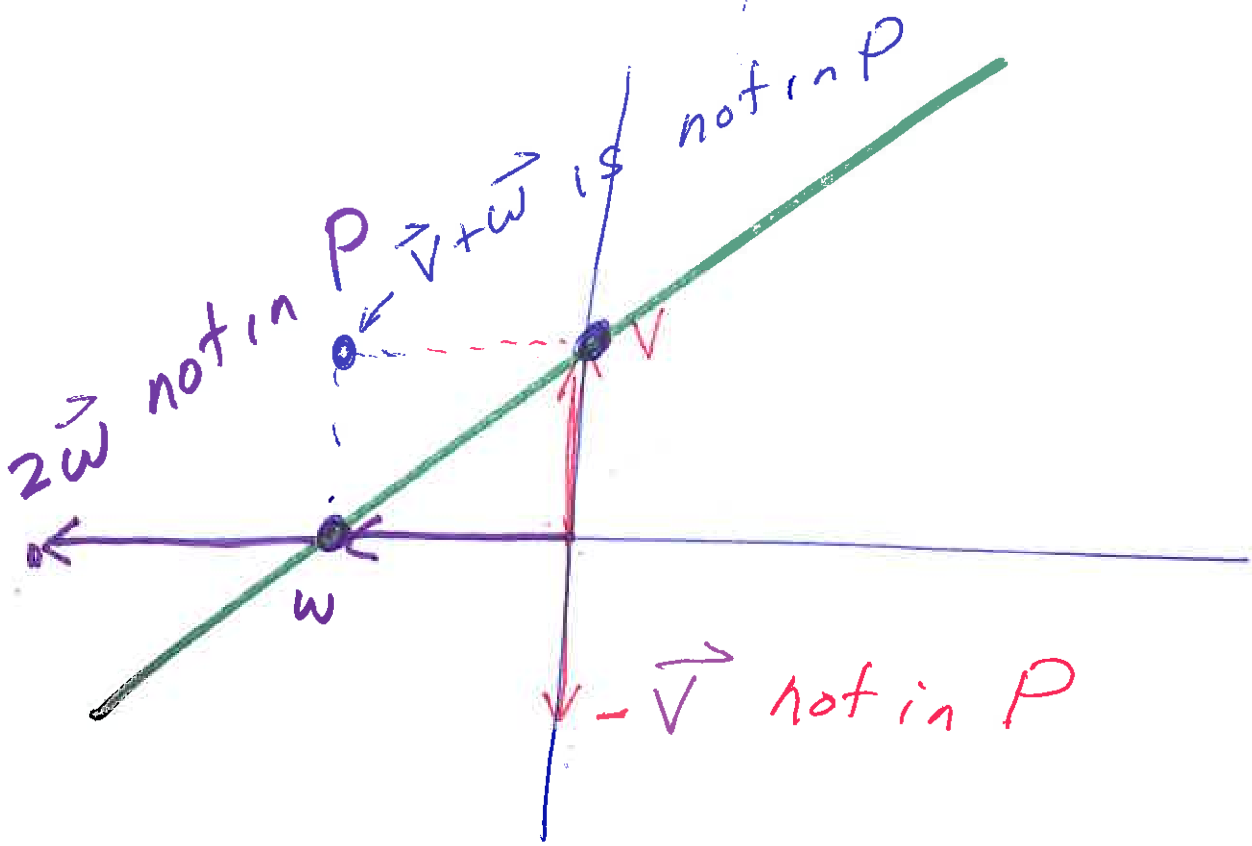
v is in P

$-v$ is not in P



Subspace

P is not a subspace



A basis is the smallest collection of vectors that spans

2.9: Basis and Dimension

not too large

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if

i.) $\mathcal{T}$ is linearly independent and

ii.) $\mathcal{T}$ spans S. large enough to span S

Examples

goldilocks approved

a.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$ is a basis for $span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$

just right

b.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix} \right\}$ is NOT a basis for $span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$

too large, not lin indep

c.) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is NOT a basis for $span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$

too small, does not span

Defn: A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. Otherwise, V is **infinite dimensional.**

Thm: All basis for a finite-dimensional vector space have the same number of elements.

Defn: $dim(V)$ = the **dimension** of a finite-dimensional vector space V = the number of vectors in any basis for S. If $dim(V) = n$, then V is said to be n-dimensional.

A basis for S is ~~the~~ a smallest collection of vectors that span S

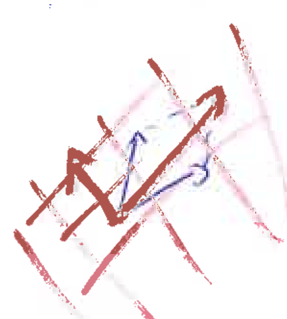
Ex: Let $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$

The following are basis

$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$

$B_2 = \left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \right\}$

$B_3 = \left\{ \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$



$$B_4 = \left\{ \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ 0 \end{bmatrix} \right\}$$

is a basis for S

Not a basis for S

$$A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$$

span $\{A\}$ is 1-dim

① A does not span S

② A not lin indep

/5

Not a basis for S

$$C = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

$$\operatorname{span} C \neq S$$

C is a basis for span C
but C is NOT a basis for S

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in C but not in S

from p 4

$S =$ xy-plane in $\mathbb{R}^3$

/6

# of elements in basis for nullspace = # of free variables

2.8 Subspaces of R^n.

Example: The **nullspace of** A is the solution set of $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

echelon form

$$\text{Nullspace of } A = \text{Solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

$$= \text{solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

EF

$$= \text{solution space of } \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

REF

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 5 & 6 & 8 & 0 \\ 3 & 7 & 9 & 12 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 - 4x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

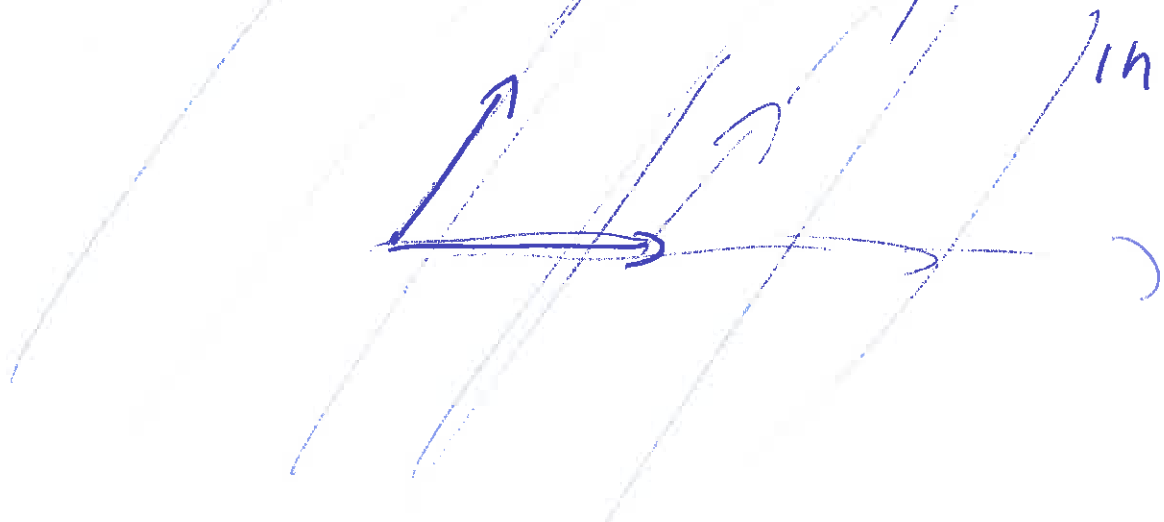
Nullspace of A

$= $ solution set $Ax = 0$

$$= \left\{ x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \begin{matrix} x_3, x_4 \\ \text{in } \mathbb{R} \end{matrix} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$=$ 2-dimensional plane living in $\mathbb{R}^4$



Examples: Nullspace and Column Space.

to find basis, remove un-needed vectors to make l.i.

Let $A = [\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}]$, a $k \times n$ matrix.

Defn: The **column space of** A = $span\{\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}\}$

Thm: The column space of A is a subspace of R^k

Note: Suppose B is row equivalent to A, then the column space of B need not be the same as the column space of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4}$$

pivot colums

free var columns

E.F.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The column space of $A = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix}}, \cancel{\begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix}} \right\}$

Took pivot columns in original matrix A

$= \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$

Thus a basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$. ■

pivot to form basis

Note we took the leading entry columns in the ORIGINAL matrix.

Why are we so interested in the column space?

Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix} x_3 + \begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix} x_4 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a sol'n?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Is $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$ = column space of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix}$?

All have the same answer

Example 1: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \\ 9 \end{bmatrix}$ have a sol'n? YES

Example 2: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 8 \\ 4 \end{bmatrix}$ have a sol'n? NO

Long method for determining IF there is a solution: ← remove extra free variable columns

$$\left[\begin{array}{cccc|cc} 1 & 2 & \cancel{4} & \cancel{3} & 9 & 3 \\ 2 & 5 & \cancel{8} & \cancel{7} & 22 & 7 \\ 3 & 7 & \cancel{12} & \cancel{8} & 31 & 8 \\ 1 & 2 & \cancel{5} & \cancel{4} & 9 & 4 \end{array}\right] \to \left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right]$$

basis for col A

Shorter method for determining IF there is a solution WHEN you know a basis for the column space:

$$\left[\begin{array}{cc|cc} 1 & 2 & 9 & 3 \\ 2^{-2} & 5^{-4} & 22^{-18} & 7^{-6} \\ 3^{-3} & 7^{-6} & 31^{-27} & 8^{-9} \\ 1^{-1} & 2^{-2} & 9^{-9} & 4^{-3} \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 9 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 1^{-1} & 4^{-4} & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

no sol'n

$$\left[\begin{array}{cc|c} 1 & 2 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

pivot columns formed basis for Col A

rank A = Rank of a matrix A = dimension of Col A

= number of pivot columns of A.

nullity of A = dimension of Nul A = number of free variables.

Solved $Ax = 0$ took basis $x_i[\ ] + x_j[\ ] + \dots$ using sol'n which depends on free variables

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Basis theorem: Let H be a $\boxed{p\text{-dimensional}}$ subspace of R^n.

i.)If $H = \text{span}\{w_1, ..., w_p\}$, then $\{w_1, ..., w_p\}$ is a basis for H.

get l.i. for free

ii.) If $v_1, ..., v_p$ are linearly independent vectors in H,

then $\{v_1, ..., v_p\}$ is a basis for H.

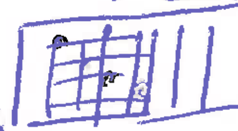
get spanning for free

$$\boxed{\textbf{Rank}(A) + \textbf{nullity}(A) = \textbf{Number of columns of } A.}$$

Ex. 1) Suppose A is a $5X\boxed{7}$ matrix.

7 columns

5 rows



If Rank$(A) = \boxed{4}$, then nullity$(A) =$ 3

$7 - 4 = 3$

$A\mathbf{x} = \mathbf{0}$ has infinite # of solutions.

5 rows 4 pivots

$A\mathbf{x} = \mathbf{b}$ has no sol'n or infinite # solutions.

RANK $A = 4$ $\Rightarrow$ 4 of 7 columns are pivot columns

$\Rightarrow$ 3 of 7 are f.v. columns

If Rank$(A) = 5$, then nullity$(A) =$

$A\mathbf{x} = \mathbf{0}$ has ______________ solutions.

$A\mathbf{x} = \mathbf{b}$ has ______________ solutions.

If Rank$(A) = 5$, the column space of $A =$

$$\left[\text{Rank}(A)\right] + \text{nullity}(A) = \#\text{ of col of } A$$

$$\text{Rank}(A) = \begin{pmatrix}\#\text{ of pivot} \\ \text{columns}\end{pmatrix} + \begin{pmatrix}\#\text{ of} \\ \text{free variabl} \\ \text{columns}\end{pmatrix} = \begin{matrix}\#\text{ of} \\ \text{col of } A\end{matrix}$$