$n \times n$

Note: In ch. 3 all matrices are $\boxed{\text{SQUARE.}}$

3.1 Defn: $detA = \Sigma \pm a_{1j_1}a_{2j_2}...a_{nj_n}$ ← FYI

$$\boxed{2 \times 2 \text{ short-cut: } det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}}$$

$$3 \times 3 \text{ short-cut: } det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \left.\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}\right\} \text{optional}$$

Note there is no short-cut for $n \times n$ matrices when $n > 3$.

Definition of Determinant using cofactor expansion

Defn: A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.

Defn: Let $A = (a_{ij})$ by an $n \times n$ square matrix. The determinant of A is

1.) If $n = 1$, $detA = a_{11}$.

2.) If $n > 1$, $detA = \Sigma_{k=1}^{n}(-1)^{1+k}a_{1k}detA_{1k}$

$$= a_{11}detA_{11} - a_{12}detA_{12} + ... + (-1)^{1+n}a_{1n}detA_{1n}$$

Note the above definition is an inductive or recursive definition.

$$\left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right.$$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$

$$- a_{31} a_{22} a_{13} - a_{32} a_{23} a_{11} - a_{33} a_{21} a_{12}$$

$$\det\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 0 & 8\end{bmatrix} = \begin{vmatrix}\cancel{1} & \cancel{2} & \cancel{3}\\4 & 5 & 6\\7 & 0 & 8\end{vmatrix}$$

$$(1)\begin{vmatrix}5 & 6\\0 & 8\end{vmatrix} - (2)\begin{vmatrix}4 & 6\\7 & 8\end{vmatrix} + (3)\begin{vmatrix}4 & 5\\7 & 0\end{vmatrix}$$

$$(-1)^{1+1}(1)\begin{vmatrix}5 & 6\\0 & 8\end{vmatrix} + (-1)^{1+2}(2)\begin{vmatrix}4 & 6\\7 & 8\end{vmatrix} + (-1)^{1+3}(3)\begin{vmatrix}4 & 5\\7 & 0\end{vmatrix}$$

$$= 1\,(5\cdot 8 - 0\cdot 6) - 2\,(\underbrace{4\cdot 8 - 7\cdot 6}_{32\ -\ 42}) + (3)(4\cdot 0 - 7\cdot 5)$$

$$= 40 + 2(+10) + 3(-35)$$

$$= 60 - 105 = -45$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$(-1)^{3+1}\, 7 \,| \quad | \quad + (-1)^{3+2}\, 0 \,| \quad | \quad + (-1)^{3+3}\, 8 \,| \quad |$$

$$+7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 8\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= 7(12-15) - 0 + 8(5-8)$$

$$= -21 + -24 = -45$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ & + & \\ & - & \end{bmatrix}$$

$$-2\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} + 0$$

$$= -2(32-42) + 5(8-21)$$

$$= 20 + 5(-13)$$

$$= 20 - 65 = -45$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & 5 & 8 \\ 2 & 0 & 6 & 9 \\ 5 & 9 & 7 & 8 \end{vmatrix}$$

$+ \quad -$

$+$

$-$

$+$

$$-2 \begin{vmatrix} \; \end{vmatrix} + 0 \begin{vmatrix} \; \end{vmatrix} - 0 \begin{vmatrix} \; \end{vmatrix} + 9 \begin{vmatrix} \; \end{vmatrix}$$

$$= -2 \begin{vmatrix} 3 & 5 & 8 \\ 2 & 6 & 9 \\ 5 & 7 & 8 \end{vmatrix} + 0 - 0 + 9 \begin{vmatrix} 1 & 3 & 4 \\ 3 & 5 & 8 \\ 2 & 6 & 9 \end{vmatrix}$$

$$-2\left[3\begin{vmatrix}6 & 9\\ 7 & 8\end{vmatrix} - 2\begin{vmatrix}5 & 8\\ 7 & 8\end{vmatrix} + 5\begin{vmatrix}5 & 8\\ 6 & 9\end{vmatrix}\right]$$

$$+9\left[1\begin{vmatrix}5 & 8\\ 6 & 9\end{vmatrix} - 3\begin{vmatrix}3 & 4\\ 6 & 9\end{vmatrix} + 2\begin{vmatrix}3 & 4\\ 5 & 8\end{vmatrix}\right]$$

$$= -2\left[3(48-63) - 2(-16) + 5(45-48)\right]$$

$$+ 9\left[(45-48) - 3(27-24) + 2(24-20)\right]$$

$$= -2\left[3(-15) + 32 - 15\right]$$

$$+ 9\left[-3 - 9 + 8\right]$$

$$= -2\left[-28\right] - 36 = 56 - 36 = 20$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} = 1 \cdot 6 \cdot 10 \cdot 2 \cdot 5 = 600$$

$$1 \begin{vmatrix} 6 & & & \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{vmatrix} - 0 + 0 - 0 + 0$$

$$= 1 \cdot 6 \left[\begin{vmatrix} 10 & 11 & 12 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{vmatrix} - 0 + 0 - 0 \right]$$

$$1 \cdot 6 \left\{ 10 \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} - 0 + 0 \right]$$

$$1 \cdot 6 \cdot 10 \left[2 \cdot 5 - 0 \right]$$

$$= 1 \cdot 6 \cdot 10 \cdot 2 \cdot 5 = 600$$

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $det A = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Cor: $det(I_n) = 1$. $det \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = 1$

Thm: If a square matrix has a row or column containing all zeros, its determinant is zero.

Thm: If some row (column) of a square matrix A is a scalar multiple of another row (column), then $det A = 0$.

Thm: A square matrix is invertible if and only if $det A \neq 0$.

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $det A \neq 0$.

Thm: $det AB = (det A)(det B)$.

Cor: $det A^{-1} = \frac{1}{det A}$.

$det(A + B) \neq det A + det B$.

Thm: $det A^T = det A$.

Thm: Let $A = (a_{ij})$ by an $n \times n$ square matrix, $n > 1$. Then expanding along row i,

$$\boxed{detA = \Sigma_{k=1}^{n} (-1)^{i+k} a_{ik} detA_{ik}.}$$

Or expanding along column j,

$$\boxed{detA = \Sigma_{k=1}^{n} (-1)^{k+j} a_{kj} detA_{kj}.}$$

choose row or column w/ most 0's

Defn: $detA_{ij}$ is the i, j-minor of A.

$\boxed{(-1)^{i+j} detA_{ij}}$ is the i, j-cofactor of A.

OR

3.2: Properties of Determinants

Thm: If $A \xrightarrow{R_i \to cR_i} B$, then $detB = c(detA)$.

Warning note: $det(cA) = c^n detA$.

Thm: If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $detB = -(detA)$.

Thm: If $A \xrightarrow{R_i + cR_j \to R_i} B$, then $detB = detA$.

CREATE ZEROS

$$\begin{vmatrix} 1 & 2 & 3 \\ 4^{-4} & 5^{-8} & 6^{-12} \\ 7^{-7} & 8^{-14} & 9^{-21} \end{vmatrix} \xrightarrow[R_3 - 7R_1]{R_2 - 4R_1} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$1 \begin{vmatrix} -3 & -6 \\ -6 & -12 \end{vmatrix}$$

$$= 1(36 - 36) = 0$$