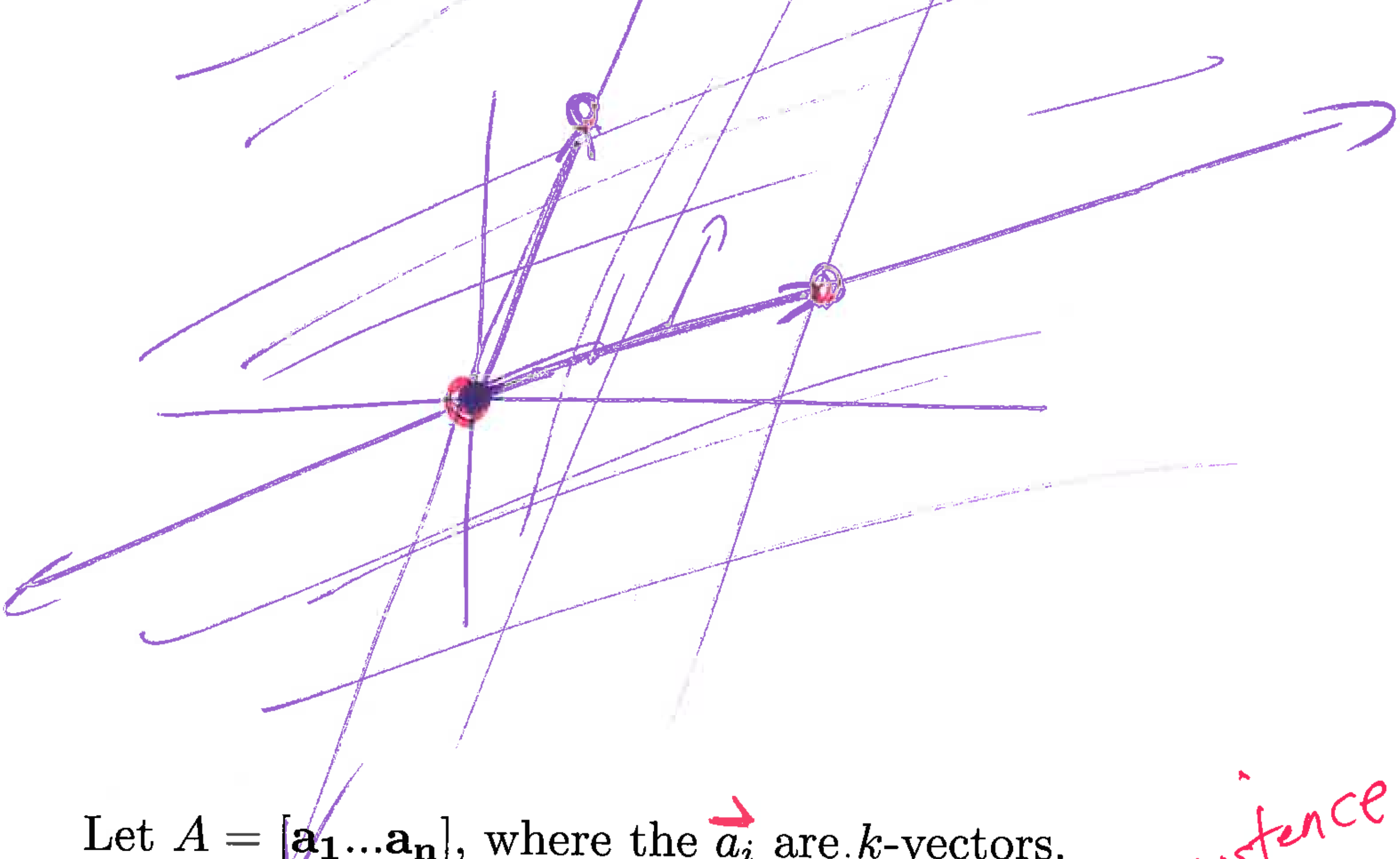
$span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ = the set of all linear combinations, $c_1\mathbf{v_1}+c_2\mathbf{v_2}+...+c_n\mathbf{v_n}$, of the vectors in $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ ∎

= the hyperplane containing the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ anchored at $\mathbf{b} = \mathbf{0}$

= the hyperplane containing the points $\mathbf{0}, \mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$



Let $A = [\mathbf{a_1}...\mathbf{a_n}]$, where the $\vec{a_i}$ are k-vectors.

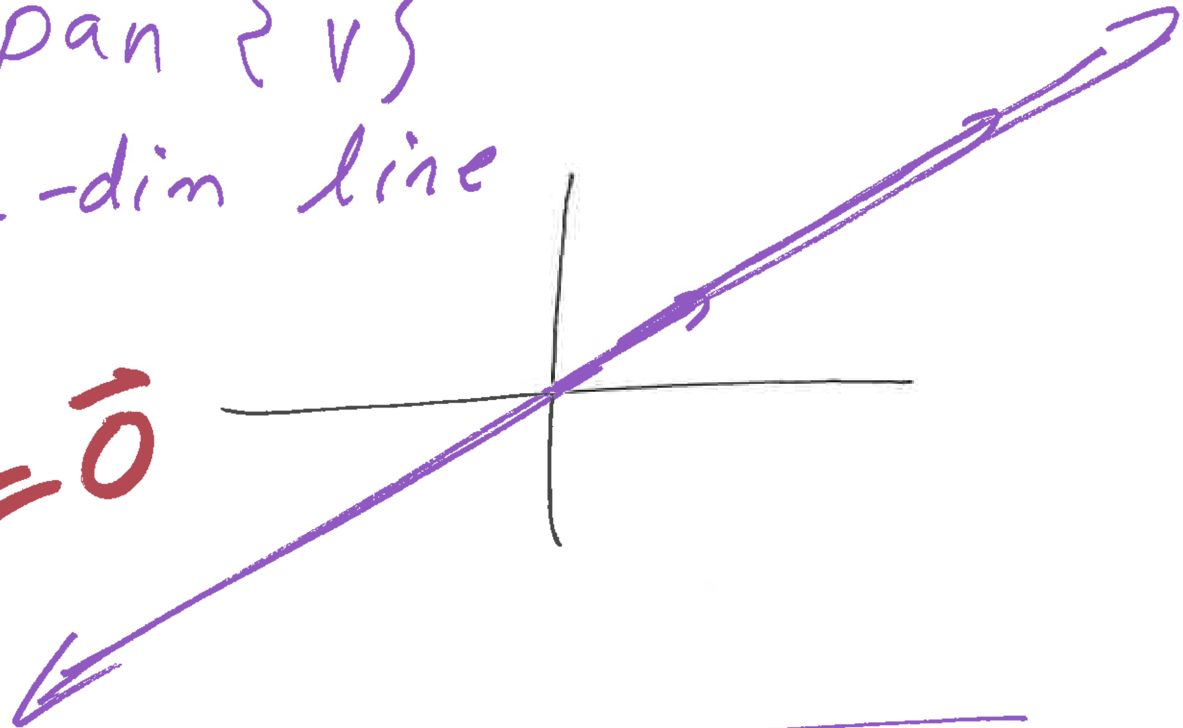
existence

1.4 $\mathbf{b}$ is in $span\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ if and only if $A\vec{x} = \vec{b}$ has at least one solution.

$span\{\mathbf{a_1}, ..., \mathbf{a_n}\} = R^k$ if and only if $A\vec{x} = \vec{b}$ has at least one solution for every $\mathbf{b}$
(leading entry in every row).

Span$\{v\}$

1-dim line



$v = \vec{0}$

Span$\{v, w\}$

2-d plane

If

$v \neq w \neq \vec{0}$ & $v \neq k w$

• $v = w = \vec{0}$

$v = k w$

Span $\{v, w\}$ is at most 2-dim

$$c_1 \vec{v} + c_2 \vec{w} = \vec{b}$$

in $\mathbb{R}^2$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} v_1 & w_1 & b_1 \\ v_2 & w_2 & b_2 \end{array}\right]$$

in $\mathbb{R}^n$

$$\left[\begin{array}{cc|c} v_1 & w_1 & b_1 \\ v_2 & w_2 & b_2 \\ \vdots & \vdots & \vdots \\ v_n & w_n & b_n \end{array}\right]$$

Does $span\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\} = R^2$? Yes, since

$$x_1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ has a sol'n for all } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

$$\text{I.e., } \begin{bmatrix} 9 & 4 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ has a sol'n for all } \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Algebraic

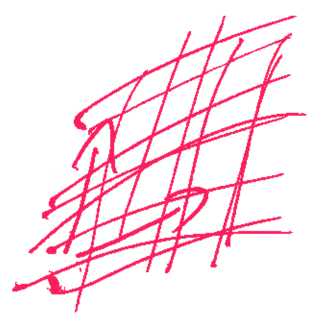
Check:

EF

$$\begin{bmatrix} 9 & 4 & b_1 \\ 7 & 8 & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 4 & b_1 \\ 0 & 8 - \frac{7}{9}(4) & b_2 - \frac{7}{9}(b_1) \end{bmatrix}$$

Thus solution exists no matter what b_1 and b_2 are.

Short-cut: $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$.



Thus span of $\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\}$ is 2-dimensional.

The only 2-dimensional plane in R^2 is R^2.

Note this short-cut only works in R^2

multiples

Does $span\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\} = R^2$? NO

Does $span\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \\ 9 \end{bmatrix} \right\} = R^4$? NO

2-dim

$3 < 4$

2-dim

Does $span\{ \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \cancel{\begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \cancel{\begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}}, \cancel{\begin{bmatrix} 10 \\ 4 \\ -3 \end{bmatrix}} \}$ [redacted] $= R^3$? NO!

$$\begin{array}{cccccc} & 3 & 2 & 4 & 7 & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} 0 & 2 & 4 & 0 & 6 & 10 & b_1 \\ 0 & 2 & 4 & -1 & 2 & 4 & b_2 \\ 0 & -3 & -6 & 2 & -1 & -3 & b_3 \end{array}\right]$$

ROW OPS

is row equivalent to

$$\underbrace{\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{coeff.}}$$

2-pivots

$\Rightarrow$ 2-dim

$$\left[\begin{array}{ccccc|c} 2 & 0 & 6 & 10 & & b_1 \\ 2 & -1 & 2 & 4 & & b_2 \\ -3 & 2 & -1 & -3 & & b_3 \end{array}\right]$$

$$\begin{bmatrix} 0 & 2 & 6 \\ -1 & 2 & 2 \\ 2 & -3 & -1 \end{bmatrix}$$

$$\downarrow$$

EF

$$\left[\begin{array}{cccccc|c} 0 & 2 & 4 & 0 & 6 & 10 & b_1 \\ 0 & 2^{-2} & 4^{-4} & -1^{-0} & 2^{-6} & 4^{-10} & b_2 - b_1 \\ 0 & -3^{+3} & -6^{+6} & 2^{+0} & -1^{+9} & -3^{+15} & b_3 + \frac{3b_1}{2} \end{array}\right]$$

$$R_2 - R_1 \longrightarrow R_3$$

$$\cancel{R_3 \longleftrightarrow R_2}$$

$$R_1/2 \longrightarrow R_1$$

$$R_3 + 3\,\text{new}\,R_1 \longrightarrow R_2$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 2 & 0 & 3 & 5 & b_1/2 \\ 0 & 0 & 0 & 2^{-2} & 8^{-8} & 12^{-12} & b_3 + \frac{3b_1}{2} + 2(b_2 - b_1) \\ 0 & 0 & 0 & -1 & -4 & -6 & b_2 - b_1 \end{array}\right]$$

$$-R_2 \longleftrightarrow R_3$$

$$R_2 + 2\,\text{old}\,R_3 \longrightarrow R_3$$

$$\left[\begin{array}{cccccc|c} 0 & 1 & 2 & 0 & 3 & 5 & b_1/2 \\ 0 & 0 & 0 & 1 & 4 & 6 & b_1 - b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_3 + \frac{3b_1}{2} + 2(b_2 - b_1) \end{array}\right]$$

This has a sol'n
if and only if

$$b_3 + \frac{3b_1}{2} + 2(b_2 - b_1) = 0$$

$$\text{span}\{ \underline{\qquad\qquad\qquad} \}$$

$$= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \;\middle|\; \underbrace{b_3 + \frac{3b_1}{2} + 2(b_2 - b_1) = 0}_{1 \text{ relation}} \right\}$$

3 variable + 1 relation

= 2-dim

$$= \left\{ \begin{bmatrix} b_1 \\ b_2 \\ -\frac{3b_1}{2} + 2(b_2 - b_1) \end{bmatrix} \;\middle|\; b_1, b_2 \in \mathbb{R} \right\}$$

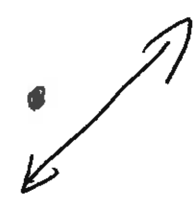
Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \cancel{\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}} \right\}$? YES

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 5 & 7 & 3 \\ 3 & 6 & 9 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -6 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$? YES

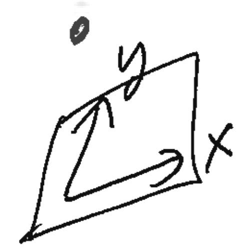
Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$? YES

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \right\}$? NO



Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$? NO

2d



Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$? YES

H - Section 1.4

$$\begin{bmatrix} a_{11} & a_{12} & \bigg| & b_1 \\ a_{21} & a_{22} & \bigg| & b_2 \end{bmatrix}$$

Augmented matrix form

solve

Solve

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

other formats

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

1.3 Linear combination format

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

1.4 Matrix format

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

From $\boxed{p5 \,\&\, 6}$ old sect 1.1, 1.2 notes
but using 1.4 notation

1.5 A system of eqns
is homogeneous if

$$A\vec{x} = \vec{0}$$

It is non-hom if

$$A\vec{x} = \vec{b}$$

$$\& \quad \vec{b} \neq \vec{0}$$

Solve:

$$\begin{aligned} 3x + 6y + 9z &= 0 \\ 4x + 5y + 6z &= 3 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

1.5: A system of equations is **homogeneous** if $b_i = 0$ for all i.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

b.) Infinite number of solutions (including, of course, $\mathbf{x} = \mathbf{0}$).

Solve:

$$\begin{aligned} 3x + 6y + 9z &= b_1 \\ 4x + 5y + 6z &= b_2 \\ 7x + 8y + 9z &= b_3 \end{aligned}$$

where 1a.) $b_1 = 0,\ b_2 = 0,\ b_3 = 0$

1b.) $b_1 = 0,\ b_2 = 3,\ b_3 = 0$

1c.) $b_1 = 6,\ b_2 = 5,\ b_3 = 8$

$$\begin{bmatrix} 3 & 6 & 9 & 0 & 0 & 6 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{bmatrix}$$

$\downarrow \frac{1}{3}R_1 \to R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{bmatrix}$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & 0 & 0 & -6 \end{bmatrix}$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

$\downarrow$ already know sol'n to system b.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow -\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \underset{R_1 - 2R_2 \to R_1}{\longrightarrow} \quad \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\vec{x} = \vec{0}$

# of solns

1) exactly 1
2) ∞ # of solns

$A\vec{x} = \vec{b}$

# of solns

1) ~~[illegible]~~ no sol'ns
2) exactly 1 soln
3) ∞ # of solns

pivot in constants column = last column of augmented matrix

Solve

$$\left[\begin{array}{ccc|c} 3 & 6 & 9 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{array}\right]$$

where

$\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|cccc} 3 & 6 & 9 & 0 & \cancel{0} & 0 & 6 \\ 4 & 5 & 6 & 0 & \cancel{0} & 3 & 5 \\ 7 & 8 & 9 & 0 & \cancel{0} & 0 & 8 \end{array}\right]$$

$$\Big\downarrow \text{ row ops}$$

no soln

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

$$A\vec{x} = \vec{0} \qquad\qquad A\vec{x} = \vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$