H - Section 1.4

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Augmented matrix form

solve

Solve

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}}$$

other formats

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

1.3 Linear combination format

$$\boxed{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}$$

1.4 Matrix format

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Matrices as linear combinations:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots, a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

linear comb

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m1} \end{bmatrix} x_2 + \dots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

ignore black bar formatting issue w/ latex

matrix format

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solve:

$$\begin{aligned} x_1 \quad + \quad\quad\quad 6x_3 &= 7 \quad -6x_3 \\ x_2 \quad + \quad 8x_3 &= 9 \quad -8x_3 \end{aligned}$$

$-6x_3$, $-8x_3$

1.1
1.2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6x_3 + 7 \\ -8x_3 + 9 \\ x_3 \end{bmatrix}$$

Solve:

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} -6x_3 \\ -8x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 7 \\ 9 \\ 0 \end{bmatrix}$$

$$= \boxed{x_3 \begin{bmatrix} -6 \\ -8 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 9 \\ 0 \end{bmatrix}}$$

MAIN SOLN FORMAT

x_3 free variable

Augmented Matrix:

$$\boxed{\left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \end{array}\right]}$$

EF → REF

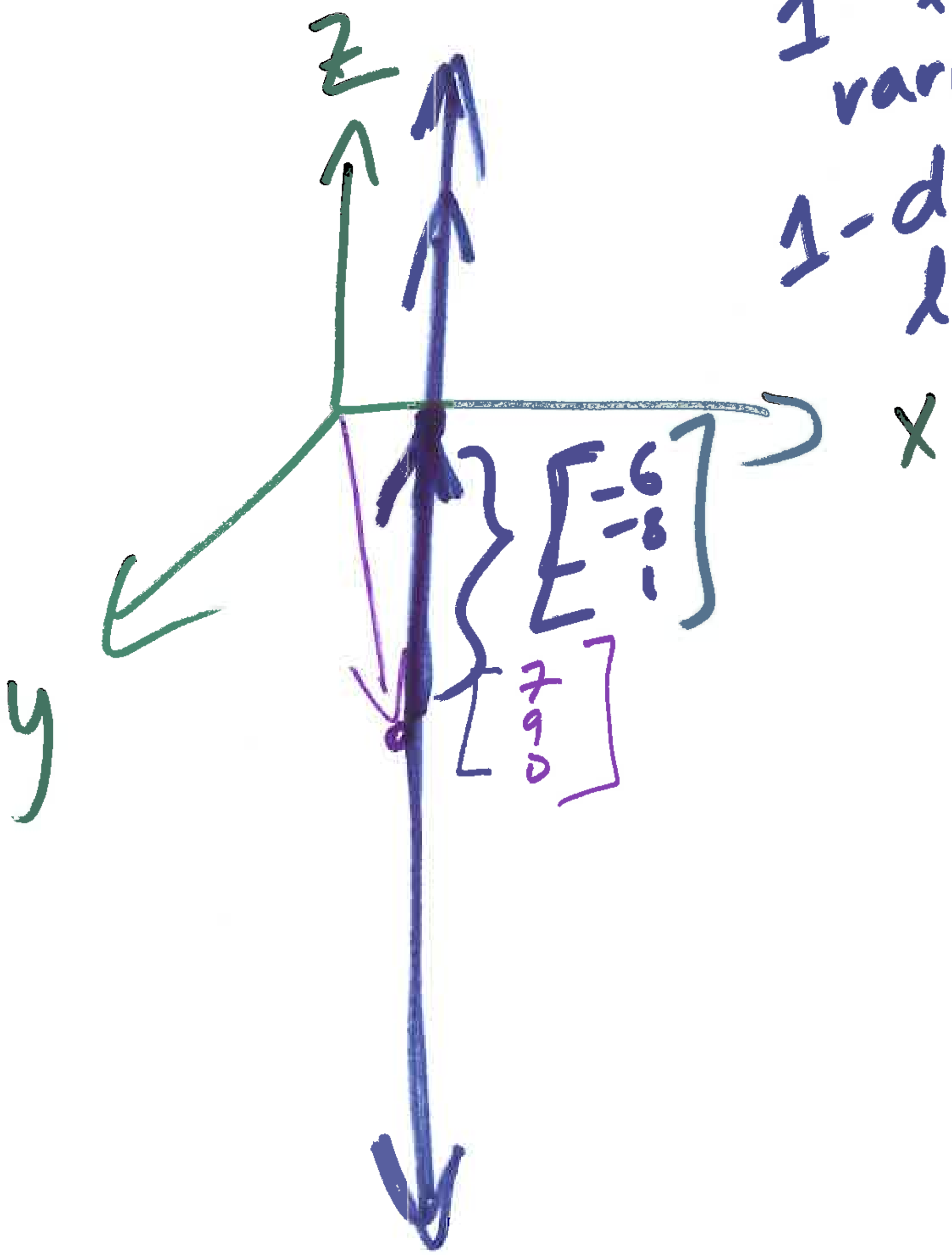
Sect 1.2

$$x_3 \begin{bmatrix}-6\\-8\\1\end{bmatrix} + \begin{bmatrix}7\\9\\0\end{bmatrix}$$

1 free variable

1-dim line

∞ sol'ns



Sect 1.3 $\begin{bmatrix}1\\0\end{bmatrix}x_1 + \begin{bmatrix}0\\1\end{bmatrix}x_2 + \begin{bmatrix}6\\8\end{bmatrix}x_3 = \begin{bmatrix}7\\9\end{bmatrix}$ AUG

- If possible write $\begin{bmatrix}7\\9\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}$

existence

$\left[\begin{array}{ccc|c}1 & 0 & 6 & 7\\0 & 1 & 8 & 9\end{array}\right]$

$$\begin{bmatrix}7\\9\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}6\\8\end{bmatrix}$$

$$\begin{bmatrix}7\\9\end{bmatrix} = 7\begin{bmatrix}1\\0\end{bmatrix} + 9\begin{bmatrix}0\\1\end{bmatrix}$$

- Is $\begin{bmatrix}7\\9\end{bmatrix}$ in $span\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\}$?

yes (since a sol'n exists)

only need EF to determine if soln exists

- Does $span\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\} = R^2$?

$$\left[\begin{array}{ccc|c}\boxed{1} & 0 & 6 & b_1\\0 & \boxed{1} & 8 & b_2\end{array}\right]$$

will there always be a sol'n

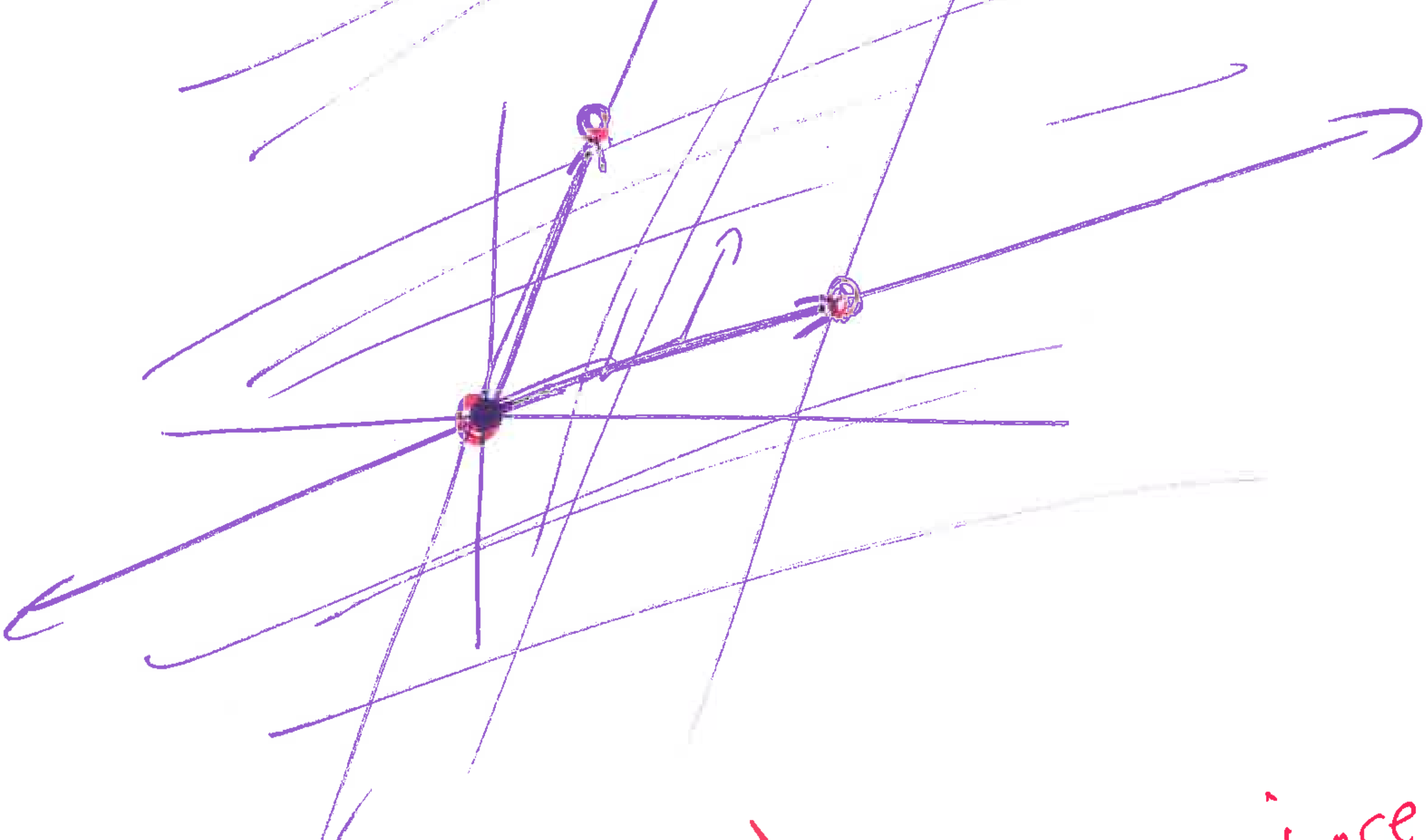
Yes

pivot in every row of COEFFICENT MATRIX (NOT AUGMENTED)

$span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ = the set of all linear combinations, $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$, of the vectors in $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ $\blacksquare$

= the hyperplane containing the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ anchored at $\mathbf{b} = \mathbf{0}$

= the hyperplane containing the points $\mathbf{0}, \mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$



Let $A = [\mathbf{a_1} ... \mathbf{a_n}]$, where the $\vec{a}_i$ are k-vectors.

existence

1.4 $\mathbf{b}$ is in $span\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ if and only if $A\vec{x} = \vec{b}$ has at least one solution.

$span\{\mathbf{a_1}, ..., \mathbf{a_n}\} = R^k$ if and only if $A\vec{x} = \vec{b}$ has at least one solution for every $\mathbf{b}$ (leading entry in every row).

Does $span\{\begin{bmatrix}9\\7\end{bmatrix}, \begin{bmatrix}4\\8\end{bmatrix}\} = R^2$? Yes, since

$x_1\begin{bmatrix}9\\7\end{bmatrix} + x_2\begin{bmatrix}4\\8\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ has a sol'n for all $\begin{bmatrix}b_1\\b_2\end{bmatrix}$.

I.e., $\begin{bmatrix}9 & 4\\7 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ has a sol'n for all $\begin{bmatrix}b_1\\b_2\end{bmatrix}$.

Algebraic

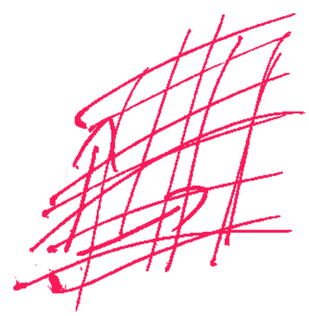
EF

Check:

$$\begin{bmatrix}9 & 4 & b_1\\7 & 8 & b_2\end{bmatrix} \rightarrow \begin{bmatrix}9 & 4 & b_1\\0 & 8-\frac{7}{9}(4) & b_2-\frac{7}{9}(b_1)\end{bmatrix}$$

Thus solution exists no matter what b_1 and b_2 are.

Short-cut: $\begin{bmatrix}4\\8\end{bmatrix}$ is not a multiple of $\begin{bmatrix}9\\7\end{bmatrix}$.



Thus span of $\{\begin{bmatrix}9\\7\end{bmatrix}, \begin{bmatrix}4\\8\end{bmatrix}\}$ is 2-dimensional.

The only 2-dimensional plane in R^2 is R^2.

Note this short-cut only works in R^2

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 8 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 4 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

$$\left[\begin{array}{cc|c} 9 & 4 & b_1 \\ 7 & 8 & b_2 \end{array}\right]$$

$$\downarrow \quad R_2 - \frac{7}{9} R_1 \to R_2$$

EF

$$\left[\begin{array}{cc|c} 9 & 4 & b_1 \\ \cancel{7 - \frac{7}{9}(9)} & 8 - \frac{7}{9}(4) & b_2 - \frac{7}{9} b_1 \end{array}\right]$$

0

not zero pivot

Sol'n exists

multiples

Does $span\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-4\\-8\end{bmatrix}\right\} = R^2$? NO

Does $span\left\{\begin{bmatrix}1\\2\\3\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\\9\end{bmatrix}\right\} = R^4$? NO

$3 < 4$

Does $span\{\overbrace{\cancel{\begin{bmatrix}0\\0\\0\end{bmatrix}}, \begin{bmatrix}2\\2\\-3\end{bmatrix}, \cancel{\begin{bmatrix}4\\4\\-6\end{bmatrix}}, \begin{bmatrix}0\\-1\\2\end{bmatrix}}^{\text{2-dim}}, \begin{bmatrix}6\\2\\-1\end{bmatrix}, \begin{bmatrix}10\\4\\-3\end{bmatrix}\} = R^3$?

$$\left[\begin{array}{cccccc|c} 0 & 2 & 4 & 0 & 6 & 10 & b_1 \\ 0 & 2 & 4 & -1 & 2 & 4 & b_2 \\ 0 & -3 & -6 & 2 & -1 & -3 & b_3 \end{array}\right]$$

is row equivalent to

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$