Ch 5 Review Questions:

$Cx = b$

$4 > 3$ more variables than ~~equations~~

3×4

# of equations   # of variables

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

$\Rightarrow$ free variables

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

0.) Does $C\mathbf{x} = \mathbf{b}$ have at most one solution for all $\mathbf{b}$?

no soln or ∞ # of soln   NO

1.) Does $C\mathbf{x} = \mathbf{0}$ have exactly one solution?

∞ # of solns   NO

2.) In an echelon form of C, is there a leading entry in every COLUMN?

NO

3.) Is $\mathbf{0}$ the only solution to $C\mathbf{x} = \mathbf{0}$?

NO

4.) Are the columns of C linearly independent?

NO

5.) Are none of the columns of C a linear comb'n of the other columns of C? NO

free variable columns are linear combinations of the other columns

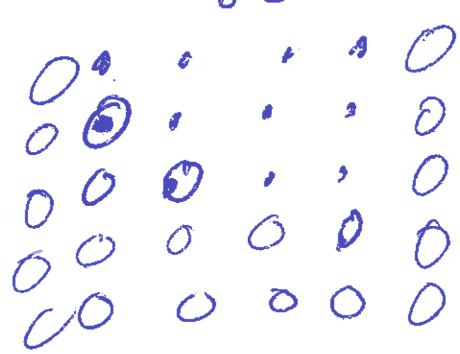
6.) Are none of the columns of C in the span of the other columns of C?

NO

$\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}$ is in span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \right\}$

more equations than variables

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = b$$



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$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Are columns Linear independent?

$$\underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 1 & 4 & 5 & 4 & 0 \\ 2 & 4 & 6 & 8 & 0 \end{array}\right]}_{\text{NO}} \sim \underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{NO}}$$

NO since we have free variables

$3 < 4$

4 vectors in $\mathbb{R}^3$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix}$$

Linear combination

If not l.i $\Rightarrow$

there exists nontrivial soln to $Cx = 0$

$$\begin{array}{cccc} & 2 & 2 & 0 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 4x_4 \\ -x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Write $\vec{0}$ as a nontrivial lin comb of the columns of C

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_4 = 0$ $\quad x_3 = 1$

$\Rightarrow x_2 = -x_3 = -1$ | $x_1 = -x_3 - 4x_4$
$= -1$

$$-\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_4 = 3$, $x_3 = 0$

$\Rightarrow x_2 = -x_3 = 0$ | $x_1 = -x_3 - 4x_4$
$= -12$

$$-12 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Trivial $x_1 = x_2 = x_3 = x_4 = 0$

Write one column as lin comb of the other columns

$$\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

see previous page

-1) $Cx = b$ never has exactly one soln

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

0.) Does $C\mathbf{x} = \mathbf{b}$ have more than one solution for some $\mathbf{b}$? YES BUT NOT FOR ALL b

∞ # of sol'ns (or no sol'n)

1.) Does $C\mathbf{x} = \mathbf{0}$ have an infinite number of solutions? YES

2.) Are there free variables in the solution to $C\mathbf{x} = \mathbf{0}$?

YES

3.) Does $C\mathbf{x} = \mathbf{0}$ have a non-zero solution? YES.

4.) Are the columns of C linearly dependent? YES

5.) Is one of the columns of C a linear comb'n of the other columns of C?

YES

6.) Is one of the columns of C in the span of the other columns of C? YES

If possible, write one of the columns of C as a linear combination of the other columns of C:

$$\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

choose x_3 & x_4 to find x_1 & x_2

use this one soln to get lin comb

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3×4

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

1.) Does $C\mathbf{x} = \mathbf{b}$ have at least one solution for all $\mathbf{b}$? NO

2.) Does $C\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$? NO

3.) In an echelon form of C, are there NO rows of all zeros? NO

4.) In an echelon form of C, is there a leading entry in every ROW? NO

5.) Can any vector in R^3 be written as a linear comb'n of the columns of C? NO

6.) Do the columns of C span R^3? NO

1b.) Find a solution to the equation $C\mathbf{x} = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$.

See scratch work at end

$$\begin{bmatrix} +2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \text{ etc}$$

2b.) Write $\begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ as a linear combination of the columns of C.

$$\begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} \text{ OR } \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

3b.) Write $3 + 7t + 6t^2$ as a linear combination of $\{1 + t + 2t^2,\ 2 + 4t + 4t^2,\ 3 + 5t + 6t^2,\ 4 + 4t + 4t^3\}$.

NOT COVERED IN 22M33

or $3 + 7t + 6t^2 = -(1 + t + 2t^2) + 2(2 + 4t + 4t^2)$

$3 + 7t + 6t^2 = 2(1 + t + 2t^2) + (2 + 4t + 4t^2) + (3 + 5t + 6t^2) - 4(4 + 4t + 4t^3)$

1a.) Does $C\mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ have at least one solution? NO

1b.) Does $C\mathbf{x} = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ have at least one solution? YES

2a.) Is $\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ a linear combination of the columns of C: NO

2b.) Is $\begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ a linear combination of the columns of C: YES

3a.) Is $4 + 2t$ a linear combination of
$\{1 + t + 2t^2,\ 2 + 4t + 4t^2,\ 3 + 5t + 6t^2,\ 4 + 4t + 4t^3\}$? NO

NOT in 22M33

3b.) Is $3 + 7t + 6t^2$ a linear combination of
$\{1 + t + 2t^2,\ 2 + 4t + 4t^2,\ 3 + 5t + 6t^2,\ 4 + 4t + 4t^3\}$? YES

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

$$\left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 4 & 3 \\ 1^{-1} & 4^{-2} & 5^{-3} & 4^{-4} & 2^{-4} & 7^{-3} \\ 2^{-2} & 4^{-4} & 6^{-6} & 8^{-8} & 0^{-8} & 6^{-6} \end{array}\right]$$

no soln
not in
span

$R_2 - R_1 \to R_2$

$R_3 - 2R_1 \to R_3$

$$\left[\begin{array}{cccc|cc} 1 & 2^{-2} & 3^{-2} & 4 & \cancel{4} & 3^{-4} \\ 0 & 2 & 2 & 0 & \cancel{-2} & 4 \\ 0 & 0 & 0 & 0 & \cancel{-8} & 0 \end{array}\right]$$

$R_1 - R_2 \to R_1$

$R_2/2 \to R_2$

no soln
so not in
span

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 4x_4 - 1 \\ -x_3 + 2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Ex of solns

Let $x_3 = x_4 = 0$

$$\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 = 1,\ x_4 = -1$

$$\begin{bmatrix} +2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$