2.1 cont: Note

$$\boxed{AB \neq BA}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

$\neq$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

It is also possible that $\boxed{AB = AC\text{, but } B \neq C.}$

$BA = CA$

In particular it is possible for $\boxed{AB = 0\text{, but } A \neq 0}$ AND $B \neq 0$

Defn: If A is a square $(n \times n)$ matrix, $A^0 = I$, $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^1 = A$, $A^k = AA \ldots A$.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

The transpose of the $m \times n$ matrix $A = A^T = (a_{ji})$.

$A = (a_{ij})$

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Transpose Properties:

a.) $(A^T)^T = A$ b.) $(A+B)^T = A^T + B^T$

c.) $(kA)^T = kA^T$ d.) $(AB)^T = B^T A^T$

Thm 1 (Properties of matrix arithmetic) Let A, B, C [redacted] be matrices. Let a, b be scalars. Assuming that the following operations are defined, then

a.) $A + B = B + A$

$\boxed{a_{ij} + b_{ij}}$

b.) $A + (B + C) = (A + B) + C$

c.) $A + 0 = A$

d.) $A + (-A) = 0$

e.) $A(BC) = (AB)C$

f.) $AI = A, IB = B$

matrices

g.) $A(B + C) = AB + AC$,
$(B + C)A = BA + CA$

real #

h.) $a(B + C) = aB + aC$

i.) $(a + b)C = aC + bC$

j.) $(ab)C = a(bC)$

k.) $a(AB) = (aA)B = A(aB)$

l.) $1A = A$

Defn.) $-A = -1A$

Cor.) $A0 = 0, 0B = 0$

matrix

Cor.) $a0 = 0$

$AB \neq BA$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Only some matrices are invertibl

2.2:

Defn: A is invertible if there exists a matrix B such that $AB = BA = I$, and B is called the inverse of A. If the inverse of A does not exist, then A is said to be singular.

Note that if A is invertible, then A is a $\boxed{\text{square}}$ matrix.

$$\underset{m \times n}{A} \quad \underset{n \times k}{B} = \underset{n \times k}{B} \quad \underset{m \times n}{A}$$

$$= I_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad k = m$$

Thm: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$, in which case

$$\boxed{A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} =$$

Ex: The inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\frac{1}{(1)(4)-(3)(2)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

since

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ \hline 3 & 4 \end{bmatrix} \underbrace{\left[\begin{array}{c|c} 4 & -2 \\ -3 & 1 \end{array}\right]}_{-2} = \underbrace{\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}}_{-2}$$

$$\begin{gathered} \| \\ 4 - 6 = ad - bc \end{gathered}$$

$$A^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Thm: Let A be a $\boxed{\text{square}}$ matrix. If there exists a square matrix B such that $AB = I$, then $BA = I$ and thus $B = A^{-1}$

If inverse exists $\Rightarrow$ unique inverse

Thm: If A is invertible, then its inverse is unique.

Proof: Suppose $AB = I$ and $CA = I$. Then, $B = IB = (CA)B = CI = C$.

Defn: $A^0 = I$, and if n is a positive integer $A^n = AA \cdots A$ and $A^{-n} = A^{-1}A^{-1} \cdots A^{-1}$.

Thm: If r, s integers, $A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Thm: If A^{-1} and B^{-1}exist, then

i.) AB is invertible and $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

$B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

ii.) A^{-1} is invertible and $\boxed{(A^{-1})^{-1} = A}$

$A^{-1}A = I = AA^{-1}$

iii.) A^r is invertible and $(A^r)^{-1} = (A^{-1})^r$

where r is any integer

iv.) For any nonzero scalar k,

kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

v.) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$.

Long method: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} =$

$$\begin{bmatrix} 2x_{11} + 3x_{21} + 4x_{31} & 2x_{12} + 3x_{22} + 4x_{32} & 2x_{13} + 3x_{23} + 4x_{33} \\ 4x_{11} + 5x_{21} + 6x_{31} & 4x_{12} + 5x_{22} + 6x_{32} & 4x_{13} + 5x_{23} + f x_{33} \\ 6x_{11} + 7x_{21} + 9x_{31} & 6x_{12} + 7x_{22} + 9x_{32} & 6x_{13} + 7x_{23} + 9x_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So solve,

$$\begin{array}{c} \begin{array}{cccc} x_{11} & x_{21} & x_{31} & \end{array} \\ \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{array}\right] \end{array} \text{ for } x_{11}, x_{21}, x_{31}.$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{array}\right] \text{ for } x_{12}, x_{22}, x_{32}.$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{array}\right] \\ \begin{array}{cccc} X_{13} & X_{23} & X_{33} & \end{array} \end{array} \text{ for } x_{13}, x_{23}, x_{33}.$$

Or shorter method, solve

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right] = [A \mid I]$$

$$\downarrow (R_2 - 2R_1 \to R_2, R_3 - 3R_1 \to R_3)$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\downarrow (-R_2 \to R_2)$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\downarrow (R_3 + 2R_2 \to R_3)$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\downarrow (R_1 - 4R_3 \to R_1, R_2 - 2R_3 \to R_2)$$

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & -3 & 8 & -4 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\downarrow (R_1 - 3R_2 \to R_1)$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -3 & -1 & 2 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \xrightarrow[(\frac{1}{2}R_1 \to R_1)]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

REF

NOTE
A is invertible
$\iff$
A is row equiv to I

unique sol'n
$\Rightarrow$ pivot in every column
$\Rightarrow$ I
since A square

$[I \mid A^{-1}]$

Thus $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{array}\right]$ is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$,

so $(x_{11}, x_{21}, x_{31}) = (-\frac{3}{2}, 0, 1)$.

$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{array}\right]$ is row equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$,

so $(x_{12}, x_{22}, x_{32}) = (\frac{-1}{2}, 3, -2)$.

$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{array}\right]$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$,

so $(x_{13}, x_{23}, x_{33}) = (1, -2, 1)$.

Shortest method:

Note that if $[A|I]$ is row equivalent to $[I|B]$, then $B = A^{-1}$.

Thus the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$ is $\begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Check answer: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

WHEN DOES A^{-1} EXIST? A is square

& A is row equivalent to I

identity matrix

ie pivot in every column of square EF of coef matrix

Solve $2x + 3y + 4z = 0$
$4x + 5y + 6z = 0$
$6x + 7y + 9z = 0$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

since $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$ is invertible $\Rightarrow$ pivot in every column $\Rightarrow$ unique sol'n

Solve $2x + 3y + 4z = 0$
$4x + 5y + 6z = 2$
$6x + 7y + 9z = 1$

$$A\vec{x} = \vec{b} \Rightarrow$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b} \Rightarrow (A^{-1}A)\vec{x} = A^{-1}b$$

multiply on left for both sides of eqn

$$I\vec{x}$$

$$\boxed{\vec{x} = A^{-1}b}$$

our unique sol'n when A^{-1} exists

$$\left[\, A^{-1} \,\right] \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \left[\, A^{-1} \,\right] \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$