Solve the following systems of equations:

homog

$$\begin{array}{cc} & x_2 \\ \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \left.\begin{array}{c} 2 \\ 0 \end{array}\right] \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{array}{ccc} x_1 & x_2 & \\ \left[\begin{array}{c} 1 \\ 0 \end{array}\right. & \begin{array}{c} 2 \\ 0 \end{array} & \left|\begin{array}{c} 0 \\ 0 \end{array}\right] \end{array}$$

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

$$\cancel{0 = 0} \qquad x_2 = \text{free variable}$$

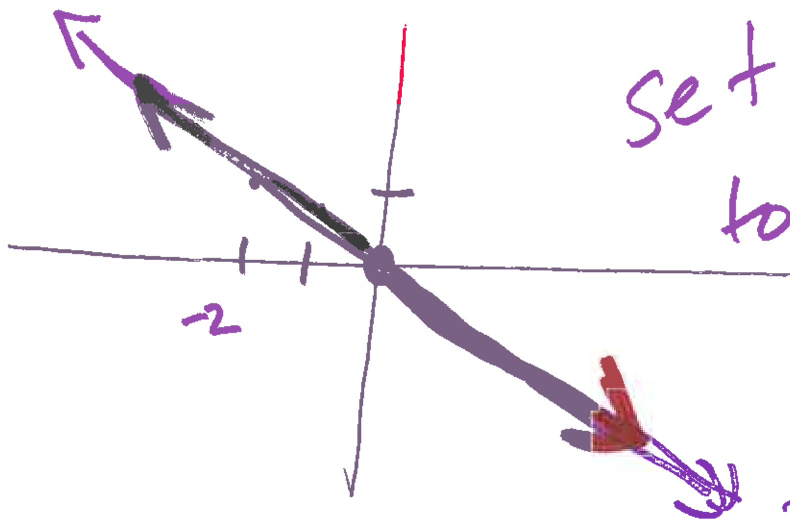
$$x_1 = -2x_2$$

$$x_2 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$



set of sol'n to homog eq w/ 1 free variable

= line thru origin

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = t\begin{bmatrix} -4 \\ 2 \end{bmatrix} = r\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

all correct answers

1.7

correct but not simplified NOT acceptable

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} + t\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

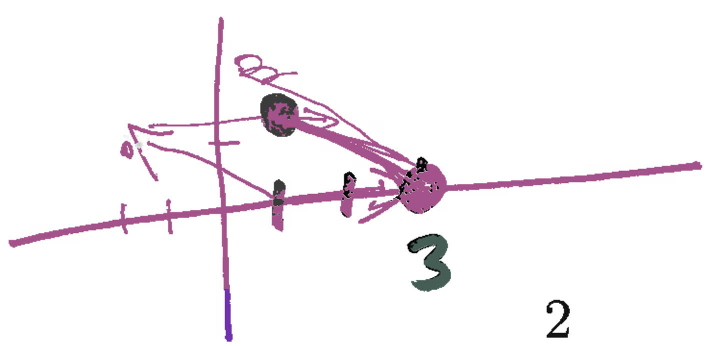
$\boxed{\text{Non-homog}}$

$$x_1 + 2x_2 = 3$$
$$0 = 0$$
$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

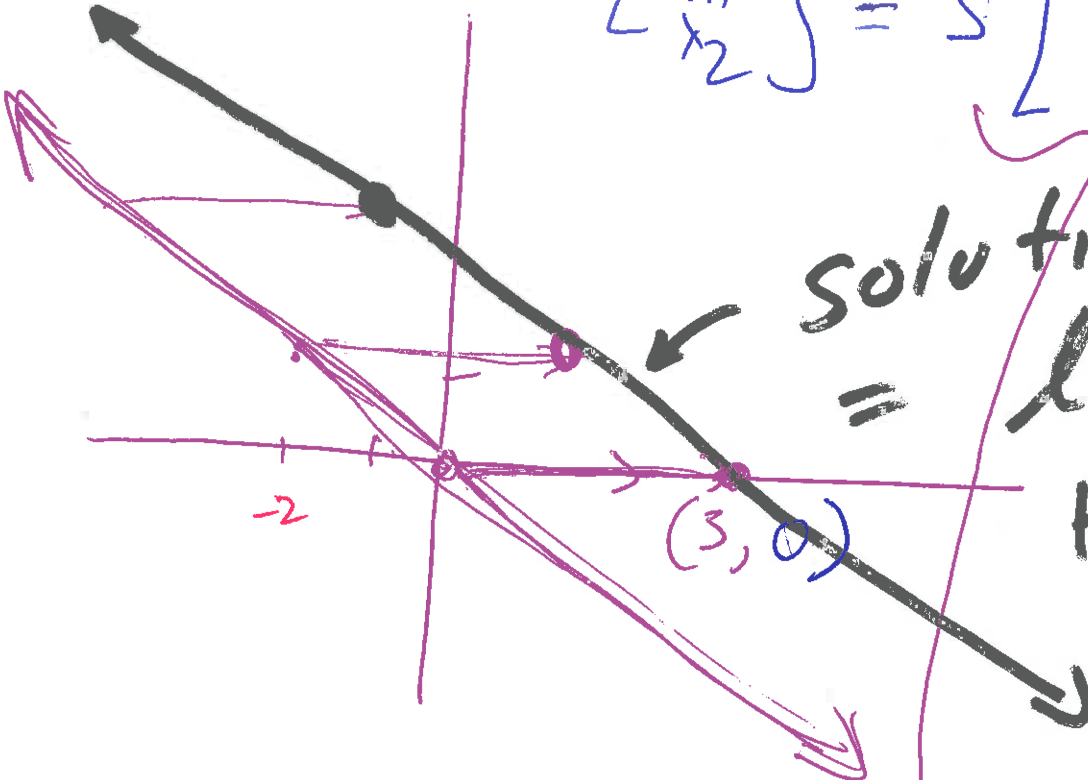
$$x_1 = -2x_2 + 3$$
$$x_2 = 1x_2 + 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3 \\ x_2 + 0 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{\text{slope } -1/2} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}$$



solution set = line thru $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ w/ slope $-1/2$

1 free variable = line

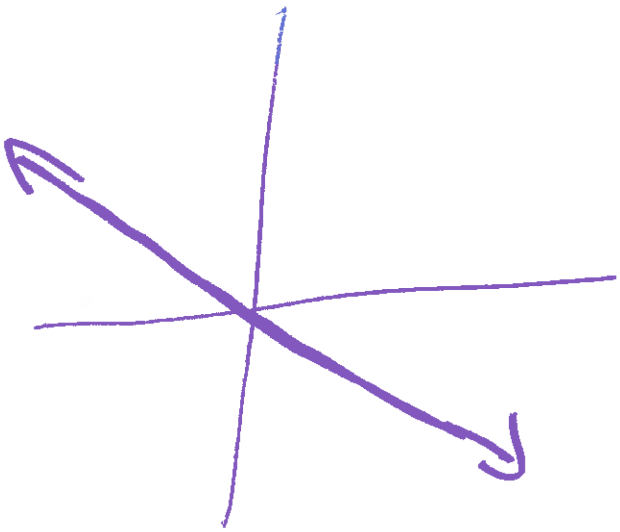
slope = same slope as homog

$Ax = b$ slope only depends on A

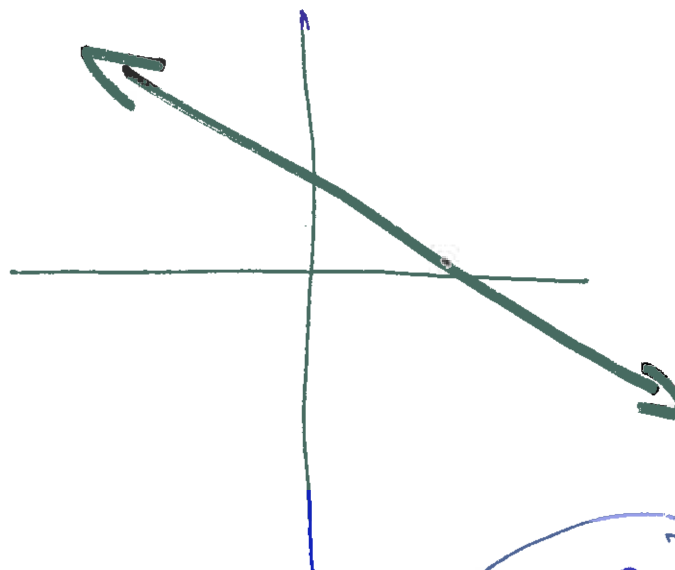
homog $\longrightarrow$ non homog

sol'n set $\longrightarrow$ translate sol'n set

$\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\vec{0}$



$\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}3\\0\end{bmatrix}$



$$\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$b_2 \neq 0$ no sol'n

If $b_2 = 0 \Rightarrow$ $$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2x_2+b_1\\x_2+0\end{bmatrix}=x_2\begin{bmatrix}-2\\1\end{bmatrix}+\begin{bmatrix}b_1\\0\end{bmatrix}$$

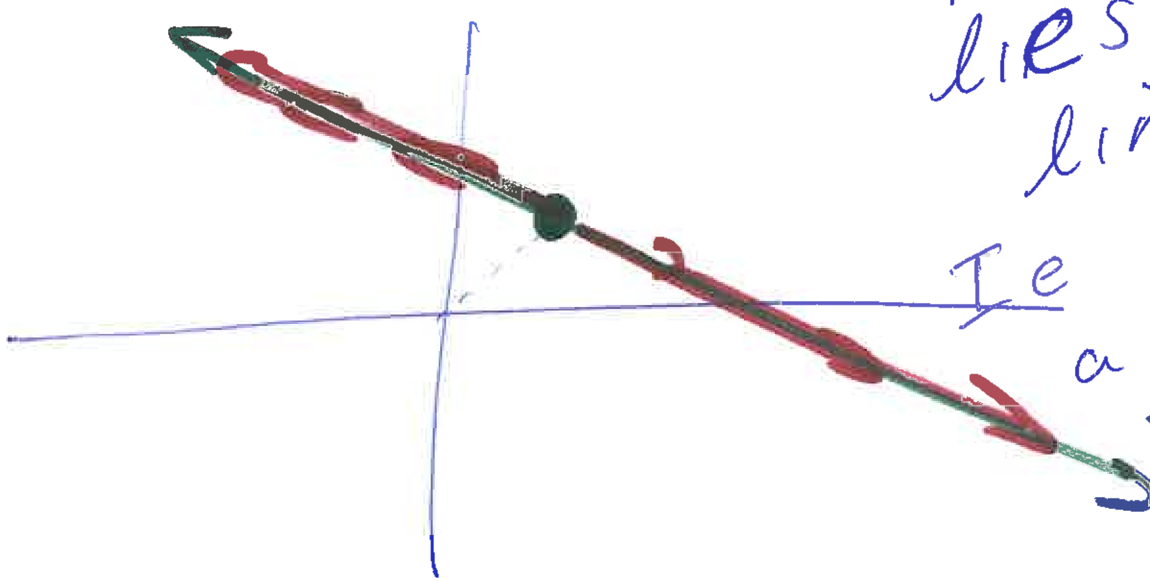
$$\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}3\\0\end{bmatrix}$$

REF $\left[\begin{array}{cc|c}1&2&3\\0&0&0\end{array}\right]$

$$\Rightarrow \vec{x}=s\begin{bmatrix}-2\\1\end{bmatrix}+\begin{bmatrix}3\\0\end{bmatrix}$$

$$=s\begin{bmatrix}-4\\2\end{bmatrix}+\begin{bmatrix}3\\0\end{bmatrix}$$

$$=s\begin{bmatrix}-2\\1\end{bmatrix}+\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\text{lies on line}}$$



is

I.e.

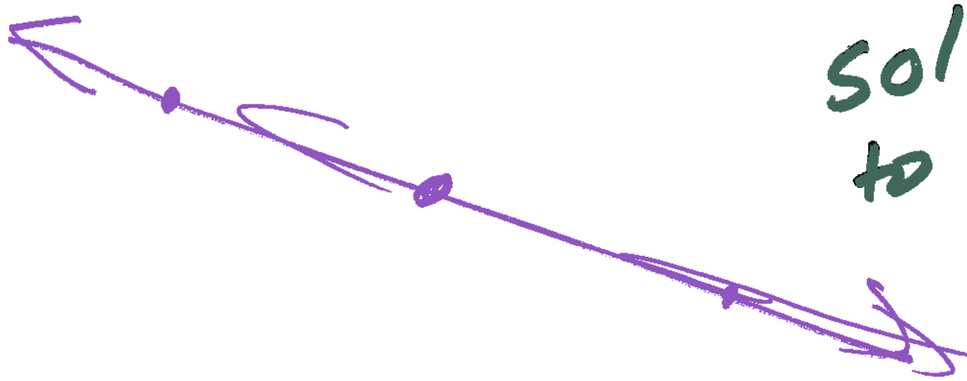
a soln to $\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}3\\0\end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\text{any sol'n to } A\vec{x} = \vec{b}}$$



$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0$$
$$0 = 0$$
$$0 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Sol'n set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

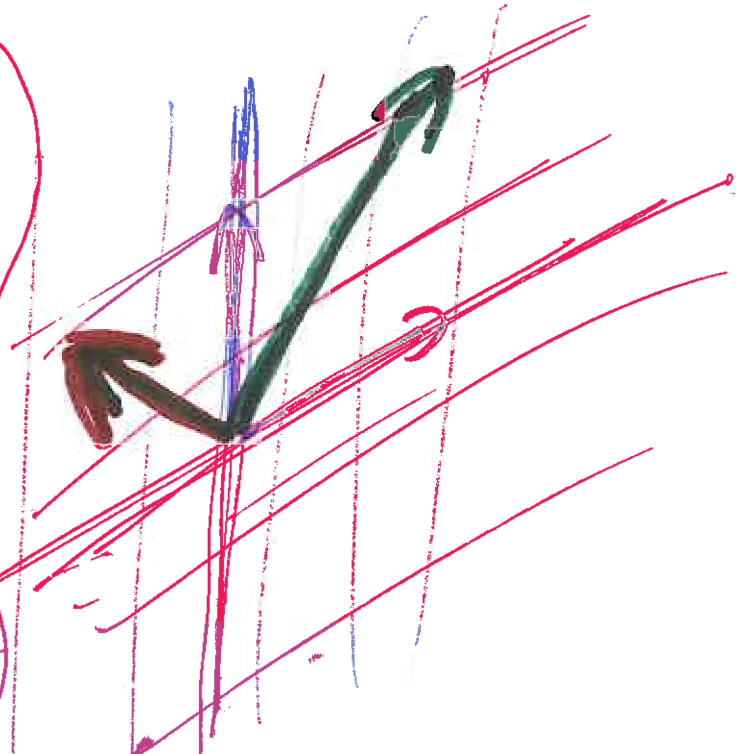
2 free variabls

$\Rightarrow$ 2-d plane

Both describle same plane

same solution set

$\Rightarrow$ Both are equally correct answers



An equivalent correct answer

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Correct but not simplified so NOT acceptable answer

$$\vec{x} = s\begin{bmatrix}-5\\1\\1\end{bmatrix} + t\begin{bmatrix}-1\\-1\\1\end{bmatrix} + u\begin{bmatrix}-2\\-2\\2\end{bmatrix}$$

$$\vec{x} = s\begin{bmatrix}-5\\1\\1\end{bmatrix} + t\begin{bmatrix}-1\\-1\\1\end{bmatrix} + \underbrace{u\begin{bmatrix}-3\\0\\1\end{bmatrix}}$$

not needed

Does not change sol'n set

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

translate homog soln by adding a ~~hom~~ sol'n to non-hom eq

A NON-homogeneous system of LINEAR equations

a.) Exactly one solution.

b.) Infinite number of solutions

c.) No solutions

A system of equations is $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = \mathbf{0}$.

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

b.) Infinite number of solutions
(including, of course, $\mathbf{x} = \mathbf{0}$).

$A\bar{x} = 0$
$\Rightarrow \bar{x} = 0$ is a soln

Solve the following systems of equations:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{bmatrix}$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & -7 & 0 & -6 \end{bmatrix}$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{bmatrix}$$

EF

no sol'n

$\downarrow$ already know sol'n to system b.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\downarrow -\frac{1}{3}R_2 \to R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{} R_1 - 2R_2 \to R_1 \quad \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3rd eqn

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xRightarrow[\text{ops}]{\text{row}} \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \xRightarrow[\text{ops}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

$0 = -6$ no sol'n

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 + 1 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and $A(c\mathbf{x}) = cA\mathbf{x}$

For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(x_1+y_1) & a_{12}(x_2+y_2) \\ a_{21}(x_1+y_1) & a_{22}(x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{11}y_1 & a_{12}x_2 + a_{12}y_2 \\ a_{21}x_1 + a_{21}y_1 & a_{22}x_2 + a_{22}y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 & a_{12}x_2 \\ a_{21}x_1 & a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{11}y_1 & a_{12}y_2 \\ a_{21}y_1 & a_{22}y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

$A(cx) = cAX$

$$= \begin{bmatrix} a_{11}cx_1 & a_{12}cx_2 \\ a_{21}cx_1 & a_{22}cx_2 \end{bmatrix} = c \begin{bmatrix} a_{11}x_1 & a_{12}x_2 \\ a_{21}x_1 & a_{22}x_2 \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \blacksquare$$

Suppose $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$, and $A\mathbf{p} = \mathbf{b}$

$A(s\vec{u} + t\vec{v}) = A(s\vec{u}) + A(t\vec{v})$

$= sA\vec{u} + tA\vec{v} = \vec{0}$

If $\vec{u}$ & $\vec{v}$ are sol'ns to $A\vec{x} = \vec{0}$

homog

$\Rightarrow$ $s\vec{u} + t\vec{v}$ is also a soln to

$A\vec{x} = \vec{0}$

homog