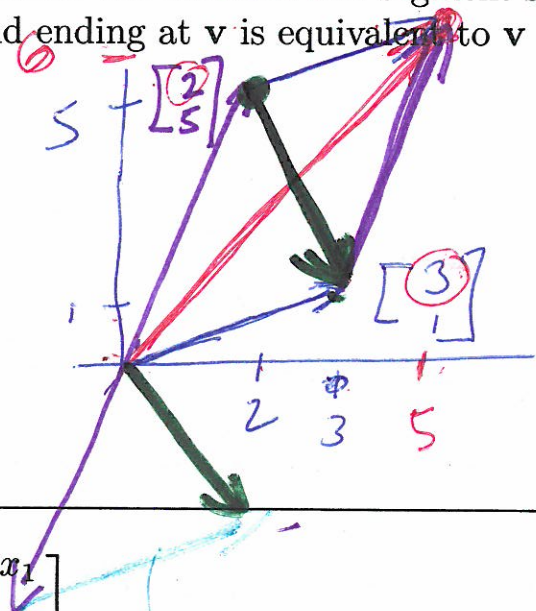
$\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}2\\5\end{bmatrix}$

$\begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}2\\5\end{bmatrix}$

Parallelogram rule:

Addition: the directed line segment starting at $\mathbf{u}$ and ending at $\mathbf{u}+\mathbf{v}$ is equivalent to $\mathbf{v}$

Subtraction: the directed line segment starting at $\mathbf{u}$ and ending at $\mathbf{v}$ is equivalent to $\mathbf{v}-\mathbf{u}$



Note $\begin{bmatrix}x_1\\ \cdot\\ \cdot\\ \cdot\\ x_n\end{bmatrix} \neq [x_1 \quad \dots \quad x_n]$

However, we will sometimes abuse notation.

Thm 3.2.1 (or thm 4.1.1 p163)

a.) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

b.) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

c.) $\mathbf{u}+\mathbf{0}=\mathbf{u}$

d.) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

e.) $(cd)\mathbf{u}=c(d\mathbf{u})$

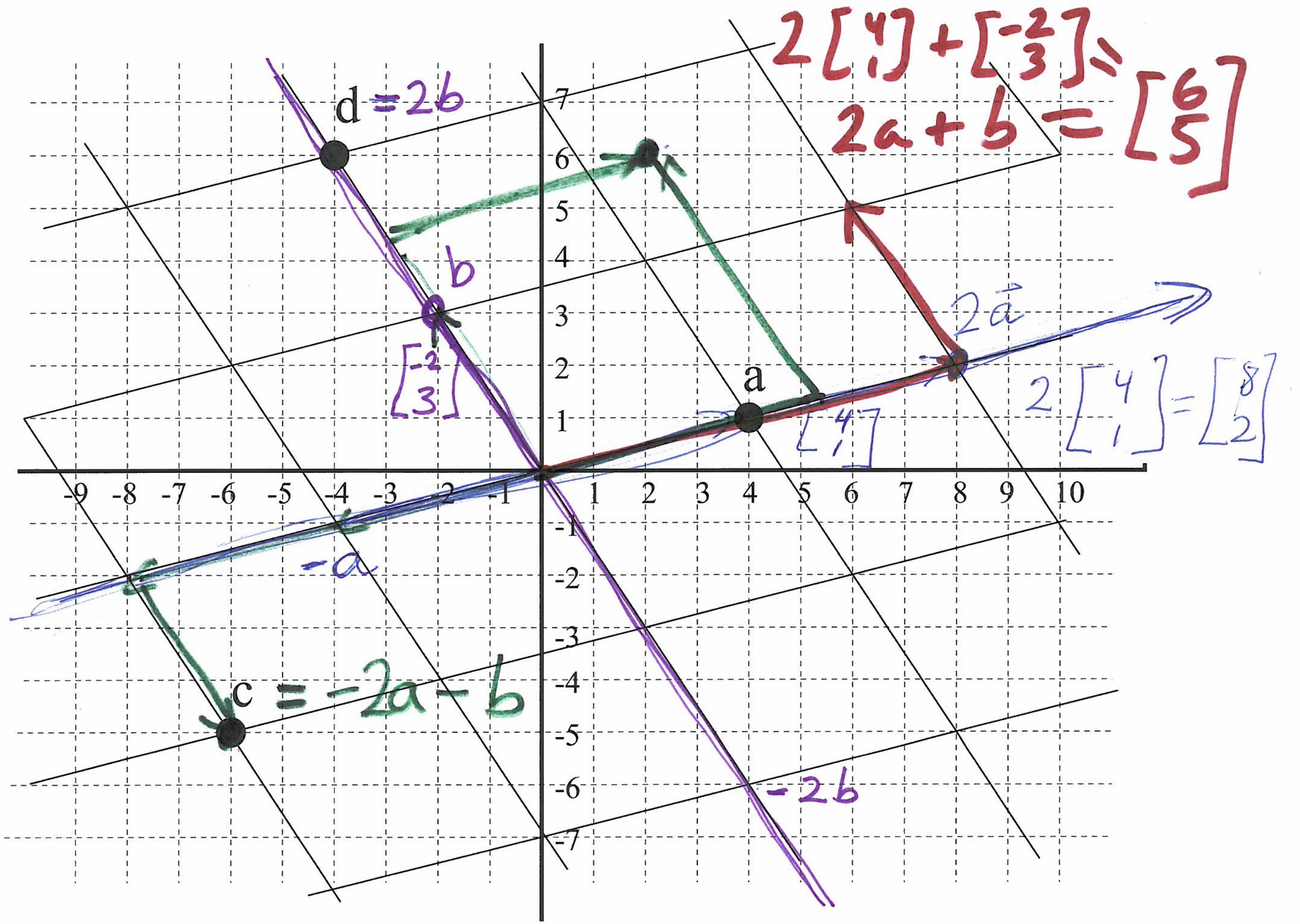
f.) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

g.) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

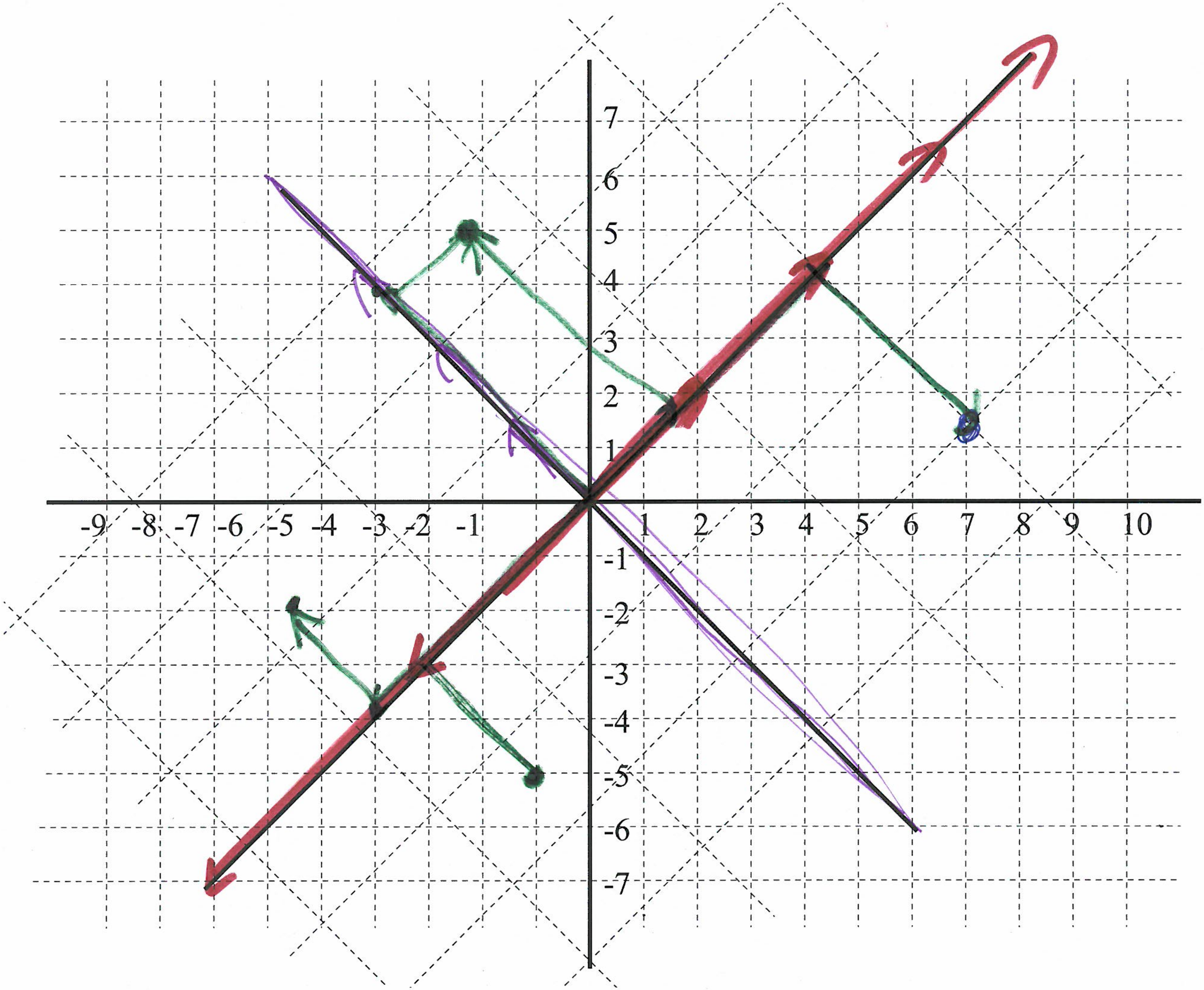
h.) $1\mathbf{u}=\mathbf{u}$

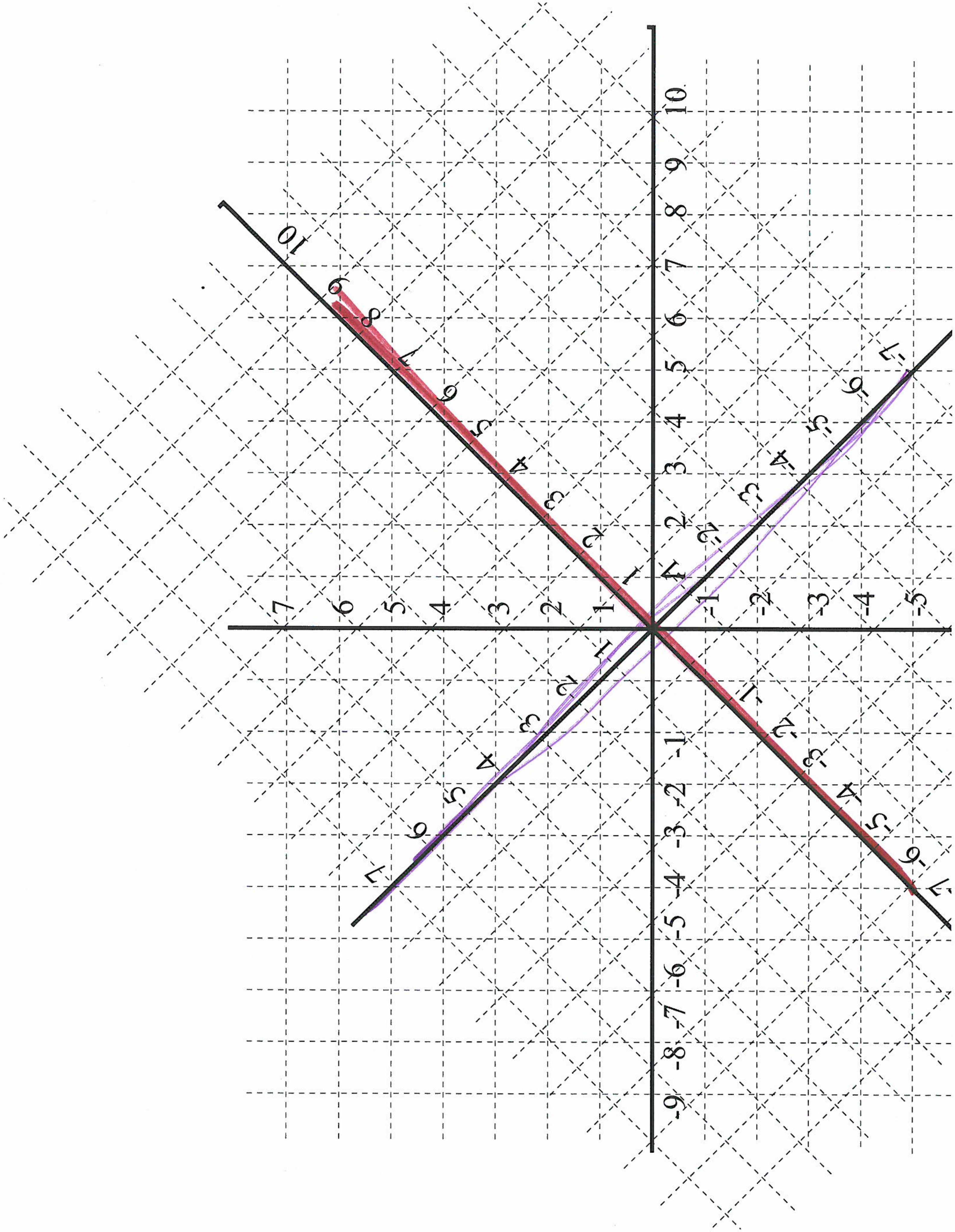
Sometimes we will write the vector $\mathbf{x}$ as a row vector: $(x_1, \dots, x_n)$.

Other times we will write the vector $\mathbf{x}$ as a column vector: $\begin{bmatrix}x_1\\ \cdot\\ \cdot\\ \cdot\\ x_n\end{bmatrix}$



$2\begin{bmatrix}4\\1\end{bmatrix}+\begin{bmatrix}-2\\3\end{bmatrix}=$
$2a+b=\begin{bmatrix}6\\5\end{bmatrix}$
d = 2b
b
$\begin{bmatrix}-2\\3\end{bmatrix}$
a
$\begin{bmatrix}4\\1\end{bmatrix}$
$2\vec{a}$
$2\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}8\\2\end{bmatrix}$
-a
c = -2a - b
-2b


-9 -8 -7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6 7 8 9 10
7 6 5 4 3 2 1 -1 -2 -3 -4 -5 -6 -7



When is $\vec{w}$ a lin comb of $\{\vec{v}_1, \ldots \vec{v}_n\}$?

If it is a l.c., find coef

1.3 Vectors in R^m

Defn: The vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ if there exist scalars $c_1, \ldots, c_n$ such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ [redacted].

$$c_1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

Does a sol'n exist?

$$\begin{bmatrix} 9c_1 \\ 7c_1 \end{bmatrix} + \begin{bmatrix} 4c_2 \\ 8c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 9c_1 + 4c_2 \\ 7c_1 + 8c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$9c_1 + 4c_2 = 3$$
$$7c_1 + 8c_2 = -5$$

$$\Rightarrow \left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7 & 8 & -5 \end{array}\right]$$

9 4 3

$7 - \frac{7}{9}(9)$ $\quad 8 - \frac{7}{9}(4)$ $\quad -5 - \frac{7}{9}(3)$

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & \frac{44}{9} & -\frac{66}{9} \end{bmatrix} \xrightarrow{\frac{9}{44}} \begin{bmatrix} 9 & 4 & 3 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$-4(1)$ $\quad -4(-\frac{3}{2})$

$$\to \begin{bmatrix} 9 & 0 & 9 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$c_1 = 1$

$c_2 = -\frac{3}{2}$

Thus, $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2)\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

If possible, write $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ [redacted]

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 9 \\ 7 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}$ [redacted]

Not possible

all l.c. of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $\begin{bmatrix} -4 \\ -8 \end{bmatrix}$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear comb'n of $\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -30 \\ 50 \end{bmatrix}$ [redacted]

lie on line thr $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y = 2x$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -30 \\ 50 \end{bmatrix}$$

Long method

$$\left[\begin{array}{cc|c} 3 & -30 & 3 \\ -5 & 50 & -5 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -10 & 1 \\ -5^{+5} & 50^{+50} & -5^{+5} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -10 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad x_1 = 10x_2 + 1$$

∞ # of sol'n

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = 41 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 4 \begin{bmatrix} -30 \\ 50 \end{bmatrix}$$

But all these answers are correct so choose 1

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 5 & 7 & 3 \\ 3 & 6 & 9 & 6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 2 & 5 & 7 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right]$$

If possible, write $\begin{bmatrix}0\\3\\6\end{bmatrix}$ as a l. c. of $\left\{ c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\5\\6\end{bmatrix} + c_3\begin{bmatrix}5\\7\\9\end{bmatrix}\right\}$

$$\begin{bmatrix}0\\3\\6\end{bmatrix} = 4\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} + \underbrace{0\begin{bmatrix}5\\7\\9\end{bmatrix}}_{\text{opt}}$$

If possible, write $\begin{bmatrix}1\\0\\0\end{bmatrix}$ as a l. c. of $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix}\right\}$

$\boxed{\text{Not possible}}$

$$\left[\begin{array}{ccc|c|c} 1 & 4 & 5 & 0 & 1 \\ 2 & 5 & 7 & 3 & 0 \\ 3 & 6 & 9 & 6 & 0 \end{array}\right] \to \begin{bmatrix} 1 & 4 & 5 & 0 & 1 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & -6 & -6 & 6 & -3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 4 & \cancel{5} & 0 & 1 \\ 0 & -3 & \cancel{-3} & 3 & -2 \\ 0 & 0 & \cancel{0} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 0 & * \\ 0 & 1 & 1 & -1 & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

$$\to \left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 4 & * \\ 0 & 1 & 1 & -1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right] \qquad \begin{aligned} c_1 &= -c_3 + 4 \\ c_2 &= -c_3 - 1 \\ c_3 &= c_3 \end{aligned}$$

$\boxed{\text{pivot in constant column} \iff \text{no sol'n}}$

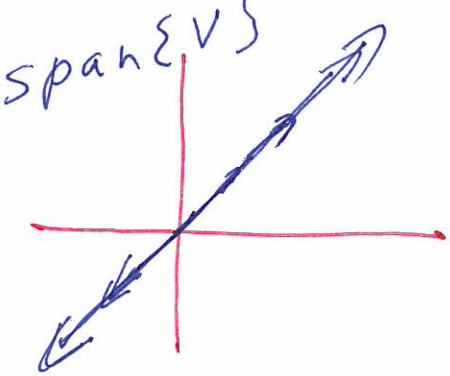
when does a soln exist

$\boxed{\begin{array}{l} span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\} = \text{the set of all linear combinations,} \\ \quad c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}, \text{ of the vectors in } \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\} \end{array}}$

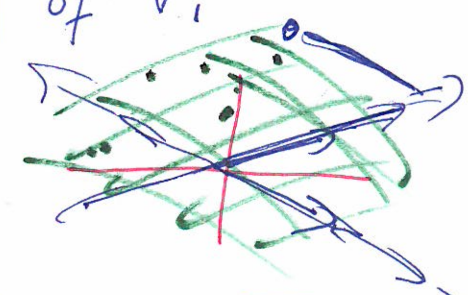
= the hyperplane containing the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ anchored at $\mathbf{b} = \mathbf{0}$

= the hyperplane containing the points $\mathbf{0}, \mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$

span{v}



span {v₁, v₂} where v₂ is NOT a multiple of v₁



Let $A = [\mathbf{a_1} ... \mathbf{a_n}]$, where the a_i are k-vectors.

$\mathbf{b}$ is in $span\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ if and only if $Ax = b$ has at least one solution.

$span\{\mathbf{a_1}, ..., \mathbf{a_n}\} = R^k$ if and only if $Ax = b$ has at least one solution for every $\mathbf{b}$ (leading entry in every row).

1.4