Math 2550 Matrix Algebra Exam 3 **Form B**
Dec. 4, 2014

[18] 1.) Find the characteristic equation and diagonalize $A = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$

NOTE: A is clearly not invertible (since det(A) = 0 or equivalently columns are linearly dependent or equivalently (since A square) rows or linearly dependent). Thus 0 is an eigenvalue of A.

Find eigenvalues:

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{vmatrix} = (2-\lambda)(8-\lambda) - 16 = 16 - 10\lambda + \lambda^2 - 36 = \lambda^2 - 10\lambda = \lambda(\lambda - 10) = 0$$

Characteristic equation of $A =$ $\underline{\lambda(\lambda - 10) = 0}$.

Find eigenvectors:

$$\lambda = 0: \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

$$\text{Check: } \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\lambda = 10: \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \text{ implies } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} x_2.$$

Thus $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is an e. vector of A. Hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is also an e-vector of A.

$$\text{Check: } \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$detP = -4 - 1 = -5$

$$P = \underline{\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}} \qquad D = \underline{\begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}} \qquad P^{-1} = \underline{\begin{bmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}}.$$

Note: if you forgot the formula for P^{-1}, you could (either derive it or) notice that A is symmetric and P is orthogonal. Thus we can normalize the columns of P so that for the new orthonormal P, $P^{-1} = P^T$ (since columns of new P are orthonormal). Thus alternative answer:

$$P = \underline{\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}} \qquad D = \underline{\begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}} \qquad P^{-1} = \underline{\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}.$$

[16] 2.) Use Gram-Schmidt to find the QR factorization of $M = \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ 4 & 6 \end{bmatrix}$

Note one can work with scaled vectors to find Q (think of the pictures relating to orthogonal projection and orthogonal component), but not R. For those not comfortable with scaling, we will work with the vectors as given.

$$\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = 4+16+16 = 36 \qquad \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = 12+0+24 = 36$$

$$proj_{\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}} \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} = \frac{36}{36} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

Orthogonal component = $\begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}$

Normalize: length of $\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \sqrt{36} = 6$ length of $\begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \sqrt{16+16+4} = 6$

Thus $Q = \begin{bmatrix} \frac{2}{6} & \frac{4}{6} \\ \frac{4}{6} & -\frac{4}{6} \\ \frac{4}{6} & \frac{2}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

$M = QR$ implies $Q^T M = Q^T QR = R$

Thus $R = Q^T M = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 4 & 0 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}+\frac{8}{3}+\frac{8}{3} & 2+0+4 \\ \frac{4}{3}-\frac{8}{3}+\frac{4}{3} & 4-0+2 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 0 & 6 \end{bmatrix}$

NOTE: Columns of Q are orthonormal and R is upper triangular.

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad R = \begin{bmatrix} 6 & 6 \\ 0 & 6 \end{bmatrix}$$

Part 2: Multiple Choice (multiple choice problems are worh 6 points each).

Problem 1. Suppose $A\begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \\ -2 \end{bmatrix}$. Then an eigenvalue of A is

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above

Problem 2. Let $A = \begin{bmatrix} -7 & -16 & 8 \\ 0 & 1 & -4 \\ 0 & 0 & -7 \end{bmatrix}$. Is A = diagonalizable?

- A. yes
- B. no
- C. none of the above

Problem 3. If x is in a subspace W, then $x - \text{proj}_W(x) = 0$.

- A. True
- B. False

Problem 4.
If the characteristic polynomial of $A = (\lambda+6)^5(\lambda-5)^2(\lambda-7)^9$, then the geometric multiplicity of $\lambda = 5$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 0 or 1
- F. 0 or 2
- G. 1 or 2
- H. 0, 1, or 2
- I. 0, 1, 2, or 3
- J. none of the above

Problem 5. The vector $\mathbf{0}$ is an eigenvector of A if and only if the columns of A are linearly dependent.

- A. True
- B. False

Problem 6. If A is symmetric if and only if A is orthogonally diagonalizable.

- A. True
- B. False

Problem 7. There are an infinite number of eigenvectors that correspond to a particular eigenvalue of A.

- A. True
- B. False

Problem 8. If the characteristic polynomial of $A = (\lambda+2)^8(\lambda+4)^2(\lambda+2)^4$, then the algebraic multiplicity of $\lambda = -4$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 0 or 1
- F. 0 or 2
- G. 1 or 2
- H. 0, 1, or 2
- I. 0, 1, 2, or 3
- J. none of the above

Problem 9. Suppose $A = PDP^{-1}$ where D is a diagonal matrix. Suppose also the d_{ii} are the diagonal entries of D. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$ and $d_{11} = d_{22}$, then $4\vec{p}_1 + 8\vec{p}_2$ is an eigenvector of A

- A. True
- B. False

Problem 10. Let $A = \begin{bmatrix} 3 & -16 & 8 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}$. Is A = diagonalizable?

- A. yes
- B. no
- C. none of the above

Problem 11.

Suppose the orthogonal projection of $\begin{bmatrix} -162 \\ 6 \\ 6 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$ is (z_1, z_2, z_3). Then z_1 =

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above