Math 2550 Matrix Algebra Exam 3 **Form A**
Dec. 4, 2014

[18] 1.) Find the characteristic equation and diagonalize $A = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$

NOTE: A is clearly not invertible (since det(A) = 0 or equivalently columns are linearly dependent or equivalently (since A square) rows or linearly dependent). Thus 0 is an eigenvalue of A.

Find eigenvalues:

$$det(A-\lambda I) = \begin{vmatrix} 20-\lambda & 10 \\ 10 & 5-\lambda \end{vmatrix} = (20-\lambda)(5-\lambda) - 100 = 100 - 25\lambda + \lambda^2 - 100 = \lambda^2 - 25\lambda = \lambda(\lambda - 25) = 0$$

Characteristic equation of $A = \underline{\lambda(\lambda-25) = 0}$.

Find eigenvectors:

$\lambda = 0$: $\begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ implies $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} x_2$

Thus $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ is an e. vector of A. Hence $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is also an e-vector of A.

$$\text{Check: } \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\lambda = 25$: $\begin{bmatrix} -5 & 10 \\ 10 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ implies $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2$.

$$\text{Check: } \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 25 \end{bmatrix} = 25 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$detP = -1 - 4 = -5$

$P = \underline{\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}}$ $\qquad D = \underline{\begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}}$ $\qquad P^{-1} = \underline{\begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}}$.

Note: if you forgot the formula for P^{-1}, you could (either derive it or) notice that A is symmetric and P is orthogonal. Thus we can normalize the columns of P so that for the new orthonormal P, $P^{-1} = P^T$ (since columns of new P are orthonormal). Thus alternative answer:

$P = \underline{\begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}$ $\qquad D = \underline{\begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}}$ $\qquad P^{-1} = \underline{\begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}}$.

[16] 2.) Use Gram-Schmidt to find the QR factorization of $M = \begin{bmatrix} 2 & 0 \\ 1 & 9 \\ 2 & 9 \end{bmatrix}$.

Note one can work with scaled vectors to find Q (think of the pictures relating to orthogonal projection and orthogonal component), but not R. For those not comfortable with scaling, we will work with the vectors as given.

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 4+1+4 = 9 \qquad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 9 \\ 9 \end{bmatrix} = 0+9+18 = 27$$

$$proj_{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 0 \\ 9 \\ 9 \end{bmatrix} = \frac{27}{9} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} 3 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

Orthogonal component = $\begin{bmatrix} 0 \\ 9 \\ 9 \end{bmatrix} - \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 3 \end{bmatrix}$

Normalize: length of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \sqrt{9} = 3$ length of $\begin{bmatrix} -6 \\ 6 \\ 3 \end{bmatrix} = \sqrt{36+36+9} = 9$

Thus $Q = \begin{bmatrix} \frac{2}{3} & -\frac{6}{9} \\ \frac{1}{3} & \frac{6}{9} \\ \frac{2}{3} & \frac{3}{9} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

$M = QR$ implies $Q^T M = Q^T QR = R$

Thus $R = Q^T M = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 9 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} \frac{4+1+4}{3} & 0+3+6 \\ \frac{-4+2+2}{3} & 0+6+3 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 0 & 9 \end{bmatrix}$

NOTE: Columns of Q are orthonormal and R is upper triangular.

$$Q = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \qquad R = \begin{bmatrix} 3 & 9 \\ 0 & 9 \end{bmatrix}$$

Part 2: Multiple Choice (multiple choice problems are worh 6 points each).

Problem 1.
If the characteristic polynomial of $A = (\lambda-4)^8(\lambda+4)^2(\lambda+7)^6$, then the geometric multiplicity of $\lambda = -4$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 0 or 1
- F. 0 or 2
- G. 1 or 2
- H. 0, 1, or 2
- I. 0, 1, 2, or 3
- J. none of the above

Problem 2. 0 is an eigenvalue of A if and only if the columns of A are linearly dependent.

- A. True
- B. False

Problem 3. Let $A = \begin{bmatrix} 7 & -1 & 3 \\ 0 & 9 & -6 \\ 0 & 0 & 7 \end{bmatrix}$. Is A = diagonalizable?

- A. yes
- B. no
- C. none of the above

Problem 4. Suppose $A = PDP^{-1}$ where D is a diagonal matrix. Suppose also the d_{ii} are the diagonal entries of D. If $P = [\ \vec{p}_1\ \vec{p}_2\ \vec{p}_3\ ]$ and $d_{11} = d_{22}$, then $3\vec{p}_1 + 4\vec{p}_2$ is an eigenvector of A

- A. True
- B. False

Problem 5. If x is in a subspace W, then $x - \text{proj}_W(x) = 0$.

- A. True
- B. False

Problem 6. If A is symmetric, then A is diagonalizable.

- A. True
- B. False

Problem 7. Suppose $A \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ 20 \\ -12 \end{bmatrix}$. Then an eigenvalue of A is

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above

Problem 8. Let $A = \begin{bmatrix} 5 & -1 & 3 \\ 0 & -6 & -2 \\ 0 & 0 & 5 \end{bmatrix}$. Is A = diagonalizable?

- A. yes
- B. no
- C. none of the above

Problem 9. The eigenspace corresponding to a particular eigenvalue of A contains an infinite number of vectors.

- A. True
- B. False

Problem 10.
If the characteristic polynomial of $A = (\lambda+7)^9(\lambda-5)^2(\lambda+3)^4$, then the algebraic multiplicity of $\lambda = 5$ is

- A. 0
- B. 1
- C. 2
- D. 3
- E. 0 or 1
- F. 0 or 2
- G. 1 or 2
- H. 0, 1, or 2
- I. 0, 1, 2, or 3
- J. none of the above

Problem 11. Suppose the orthogonal projection of $\begin{bmatrix} 92 \\ -1 \\ -6 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$ is (z_1, z_2, z_3). Then $z_1 =$

- A. -4
- B. -3
- C. -2
- D. -1
- E. 0
- F. 1
- G. 2
- H. 3
- I. 4
- J. none of the above