1. (1 pt) local/Library/UI/Fall14/quiz2_9.pg

Supppose A is an invertible $n \times n$ matrix and v is an eigenvector of A with associated eigenvalue -5. Convince yourself that v is an eigenvector of the following matrices, and find the associated eigenvalues:

1. A^8, eigenvalue =

$A\vec{v} = \lambda\vec{v}$

$A^2\vec{v} = A(A\vec{v})$

$= A(\lambda\vec{v})$

$= \lambda(A\vec{v})$

$= \lambda(\lambda\vec{v})$

$= \lambda^2\vec{v}$

$A^8\vec{v} = \lambda^8\vec{v}$

- A. 16
- B. 81
- C. 125
- D. 216
- E. 1024
- F. 390625
- G. 2000
- H. None of those above

2. A^{-1}, eigenvalue =

- A. -0.5
- B. -0.333
- C. -0.2
- D. -0.125
- E. 0
- F. 0.125
- G. 0.333
- H. 0.5
- I. None of those above

3. $A - 4I_n$, eigenvalue =

$(A - 4I_n)\vec{v}$

$A\vec{v} - 4I_n\vec{v}$

$= \lambda\vec{v} - 4\vec{v}$

$= (\lambda - 4)\vec{v}$

- A. -8
- B. -4
- C. -5
- D. 0
- E. 2
- F. 4
- G. -9
- H. 10
- I. None of those above

4. $8A$, eigenvalue =

- A. -36
- B. -28
- C. -40
- D. -12
- E. 0
- F. 24
- G. 36
- H. None of those above

2. (1 pt) local/Library/UI/Fall14/quiz2_10.pg

If $v_1 = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are eigenvectors of a matrix A corresponding to the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 4$, respectively, then

a. $A(v_1 + v_2) =$

$A\vec{v}_1 + A\vec{v}_2$

- A. $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$
- B. $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$
- C. $\begin{bmatrix} -6 \\ 5 \end{bmatrix}$
- D. $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$
- E. $\begin{bmatrix} 12 \\ 4 \end{bmatrix}$
- F. $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$
- G. None of those above

b. $A(-3v_1) =$

- A. $\begin{bmatrix} -12 \\ -12 \end{bmatrix}$
- B. $\begin{bmatrix} -2 \\ 8 \end{bmatrix}$
- C. $\begin{bmatrix} -6 \\ 4 \end{bmatrix}$
- D. $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$
- E. $\begin{bmatrix} 30 \\ -30 \end{bmatrix}$
- F. $\begin{bmatrix} 12 \\ 4 \end{bmatrix}$
- G. $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$
- H. None of those above

7.2: Quadratic Forms $\boxed{Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}}$ where A is symmetric.

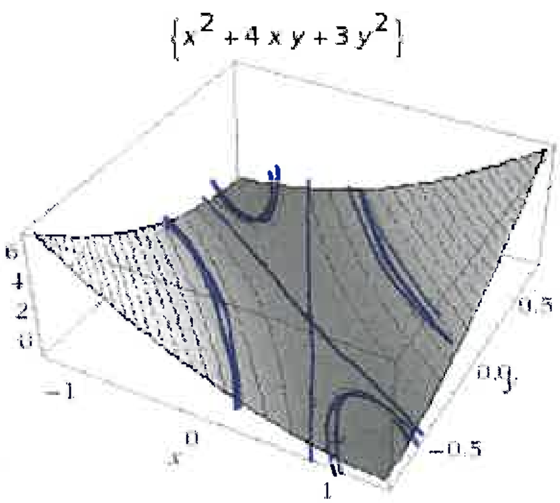
Example: $Q : R^2 \to R$

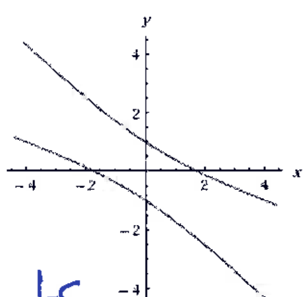
$$Q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

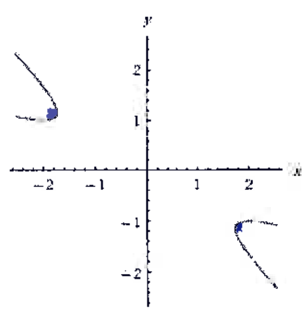
$$[x\ y] \begin{bmatrix} x+2y \\ 2x+3y \end{bmatrix} = x(x+2y) + y(2x+3y)$$

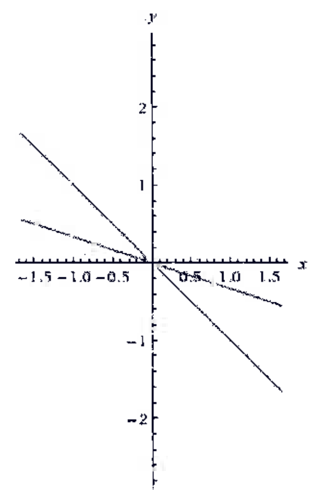
$$= x^2 + 2xy + 2xy + 3y^2$$

$$\boxed{Q(x,y) = x^2 + 4xy + 3y^2}$$









Level sets

$x^2 + 4xy + 3y^2 = 4$ $\quad$ $x^2 + 4xy + 3y^2 = -1$ $\quad$ $x^2 + 4xy + 3y^2 = 0$

$Q(x,y) = 4$ $\quad$ $Q(x,y) = -1$ $\quad$ $Q(x,y) = 0$

$$Q(x,y) = 5x^2 + 8xy + 7y^2$$

$$\div 2$$

$$= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 5 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Q(e_1) = Q(1,0) \quad \Big| \quad \vec{e}_1 = (1,0)$$

$$Q(1,0) = 5(1)^2 + 8(1)(0) + 7(0)^2$$

$$= 5 + 0 + 0 = \underline{5}$$

$$e_2 = (0,1)$$

$$Q(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 5 & 4 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 7 \end{bmatrix} =$$

1. (1 pt) **Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_5.pg**

If $A = \begin{bmatrix} 2 & -6 & 9 \\ -6 & 6 & -5 \\ 9 & -5 & -7 \end{bmatrix}$ and $Q(x) = \underline{x \cdot Ax}$, $\underline{= x^T A x}$

Then $Q(x_1,x_2,x_3) =$ ___ x_1^2+ ___ x_2^2+ ___ x_3^2+ ___ x_1x_2+ ___ x_1x_3+ ___ x_2x_3.

2. (1 pt) **Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_1.pg**

Write the matrix of the quadratic form

$Q(x) = -9x_1^2 + 2x_2^2 - 1x_3^2 - 1x_1x_2 + 3x_1x_3 - 9x_2x_3$.

$A = \begin{bmatrix} \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \\ \_\_ & \_\_ & \_\_ \end{bmatrix}$.

3. (1 pt) **Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_4.pg**

If $A = \begin{bmatrix} 2 & 8 \\ 8 & 3 \end{bmatrix}$ and $Q(x) = x \cdot Ax$, Hint: $e_1 = (1,0)$

Then $Q(e_1) =$ ___ and $Q(e_2) =$ ___. $e_2 = (0,1)$

4. (1 pt) **Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_2.pg**

Find the eigenvalues of the matrix

$M = \begin{bmatrix} -55 & 5 \\ 5 & -55 \end{bmatrix}$.

Enter the two eigenvalues, separated by a comma:

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e_2 ↑ ↳ e_1

Classify the quadratic form $Q(x) = x^T Ax$:

- A. $Q(x)$ is indefinite > 0 < 0
- B. $Q(x)$ is negative definite > 0
- C. $Q(x)$ is positive definite < 0
- D. $Q(x)$ is positive semidefinite ≥ 0
- E. $Q(x)$ is negative semidefinite ≤ 0

5. (1 pt) **Library/Rochester/setLinearAlgebra23QuadraticForms-/ur_la_23_3.pg**

The matrix

$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

has three distinct eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3$,

$\lambda_1 =$ ___,

$\lambda_2 =$ ___,

$\lambda_3 =$ ___.

Classify the quadratic form $Q(x) = x^T Ax$:

- A. $Q(x)$ is positive definite
- B. $Q(x)$ is positive semidefinite
- C. $Q(x)$ is indefinite
- D. $Q(x)$ is negative definite
- E. $Q(x)$ is negative semidefinite

More examples: $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ where A is symmetric.

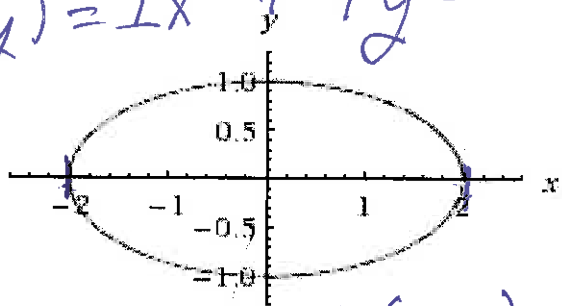
e. values $= 1, 4$

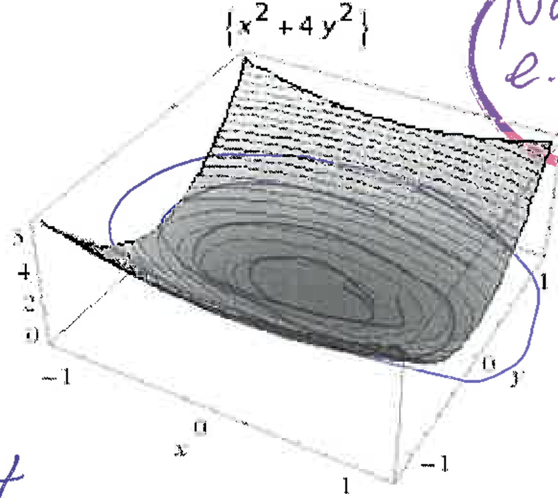
Note both e. values > 0

posit defin

$$Q(x,y) = [x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$Q(x,y) = 1x^2 + 4y^2$

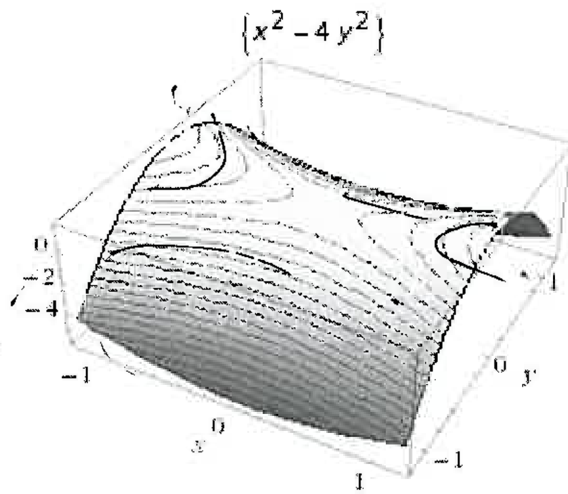




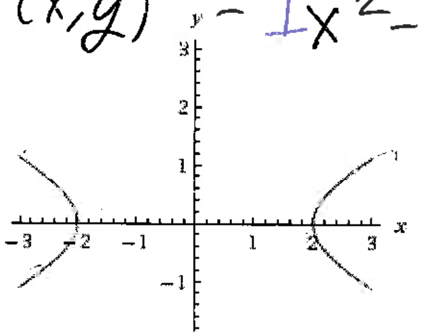
$Q(x,y) = 4$

$$Q(x,y) = [x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$Q(x,y) = 1x^2 - 4y^2$



$z_0 > 0$



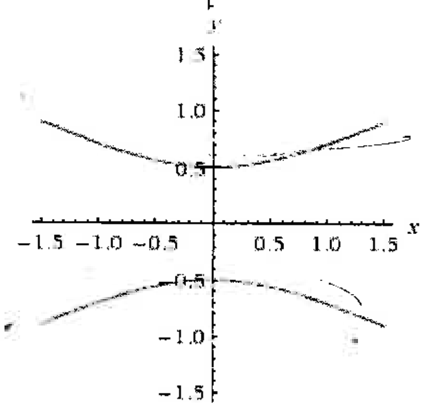
e. v $= 1, -4$

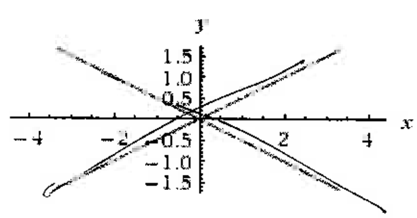
1 ev > 0

1 ev < 0

indefinite

$z_0 < 0$





$Q(x,y) = 0$

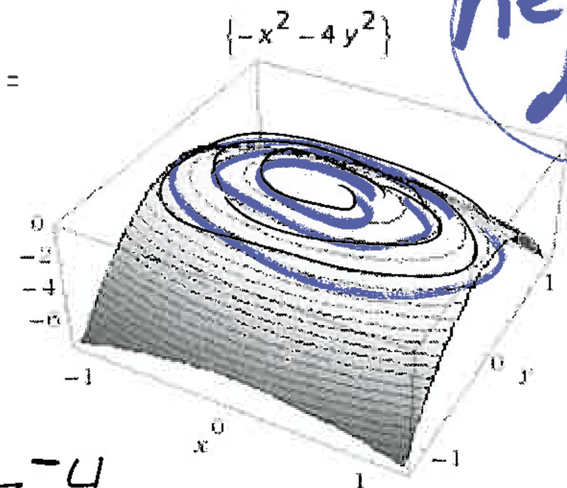
$z_0 = 0$

$Q(x,y) = -x^2 - 4y^2$

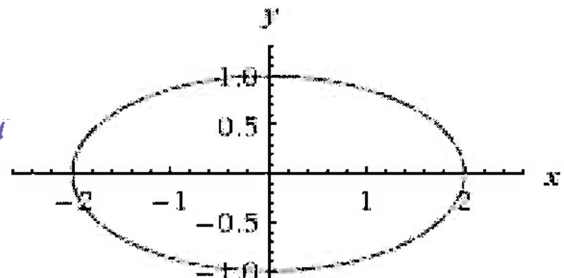
all e.v < 0

negative definite

$$Q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$



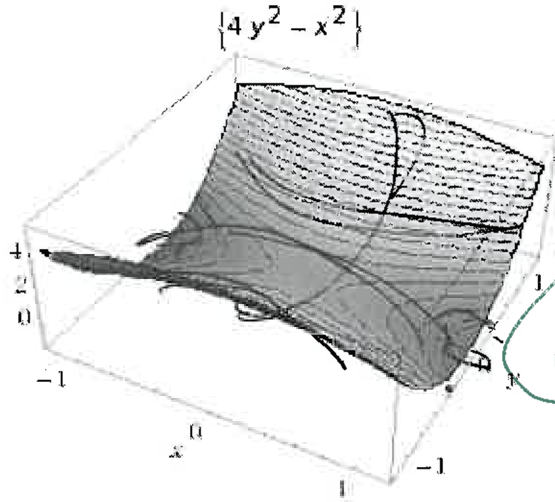
$z_0 = -4$



$Q(x,y) = -x^2 - 4y^2 = -4$

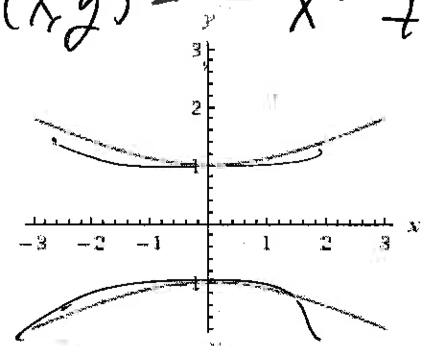
$$Q(x,y) = [x \quad y] \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$Q(x,y) = -x^2 + 4y^2$

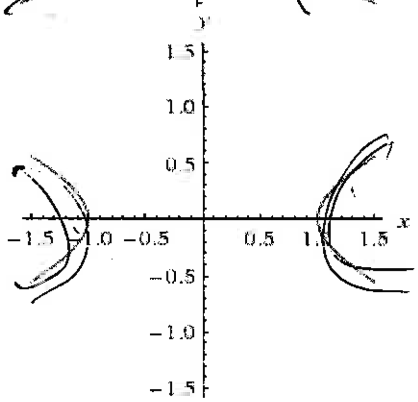


indefin

$z_0 > 0$



$z_0 < 0$

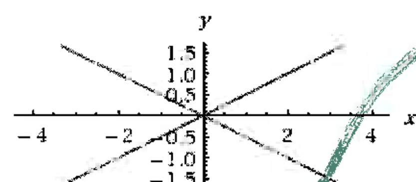


e.v $= -1, 4$

1 e.v < 0

1 e.v > 0

indefinite

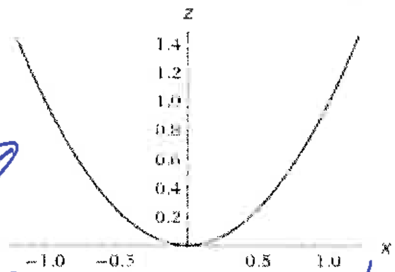


$z_0 = 0$

e. values = 0

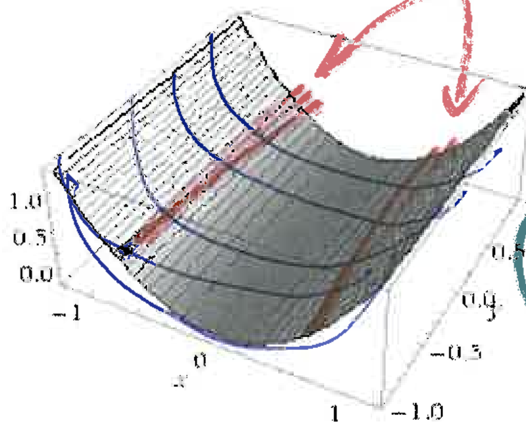
$Q(x,y) = x^2$

$$Q(x,y) = [x \quad y] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



(x from −1 to 1)

cross section



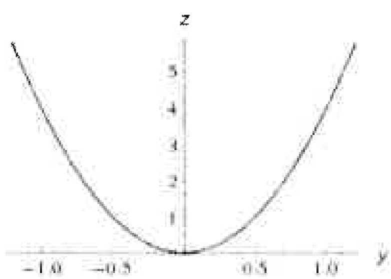
level set

ev ≥ 0

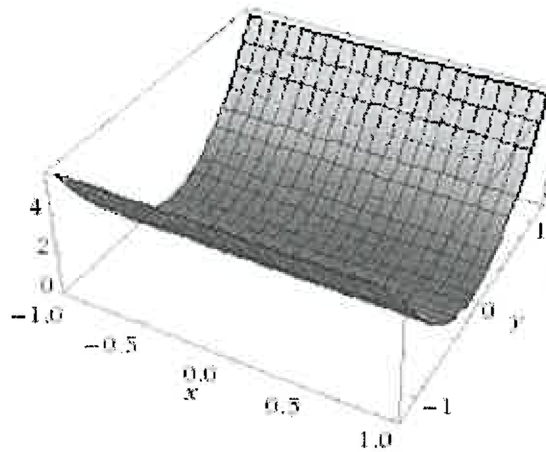
pos semi def

$Q(x,y) = 4y^2$

$$Q(x,y) = [x \quad y] \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



(y from −1 to 1)

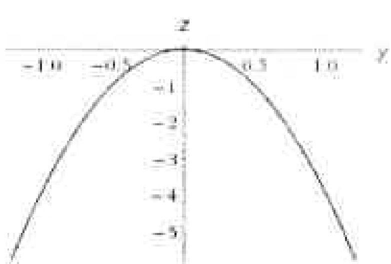


e.v ≥ 0

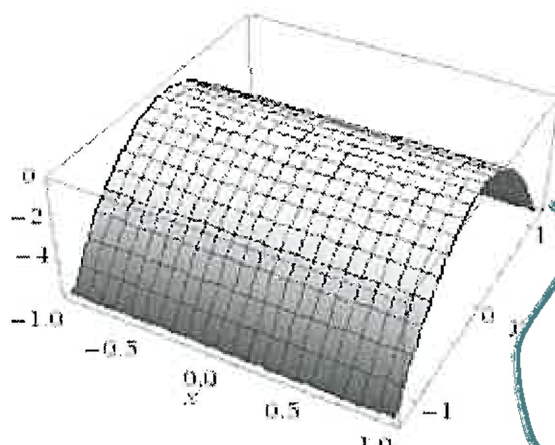
pos semi def

$Q(x,y) = -4y^2$

$$Q(x,y) = [x \quad y] \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



(y from −1 to 1)



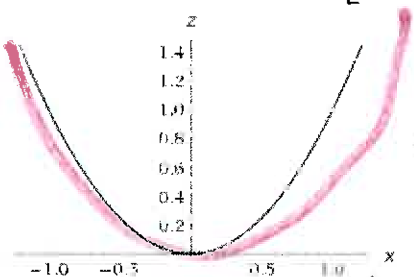
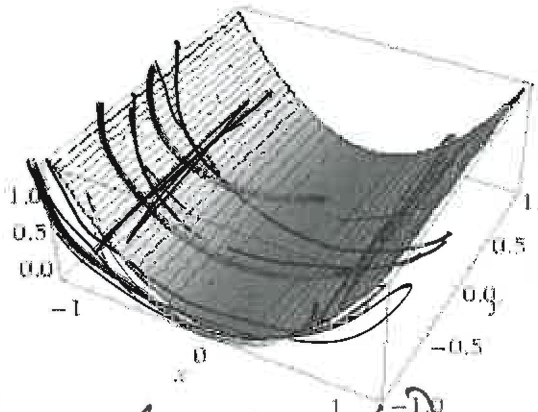
ev ≤ 0

neg semi def

Not neg def since e.v = 0

$Q(x,y) = x^2$ pos semi-def

$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

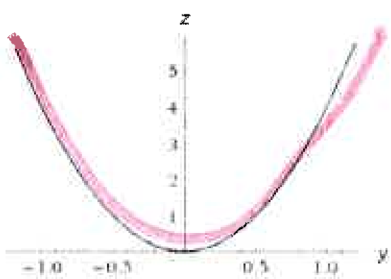
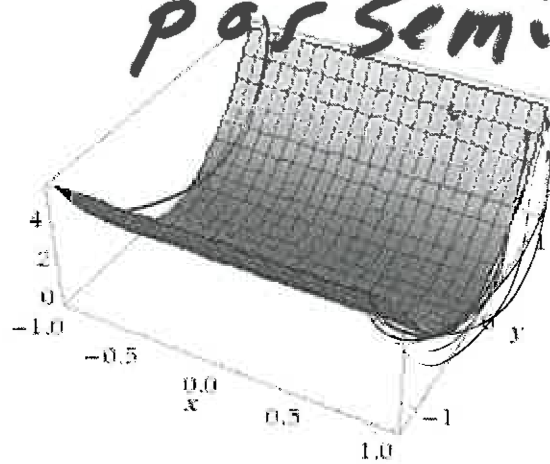
e.v = 0

e.v > 0

cross-section (not a level set)

$Q(x,y) = 4y^2$ pos semi def

$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

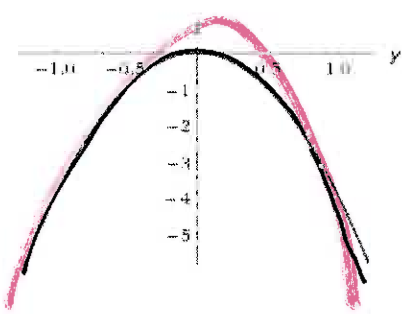
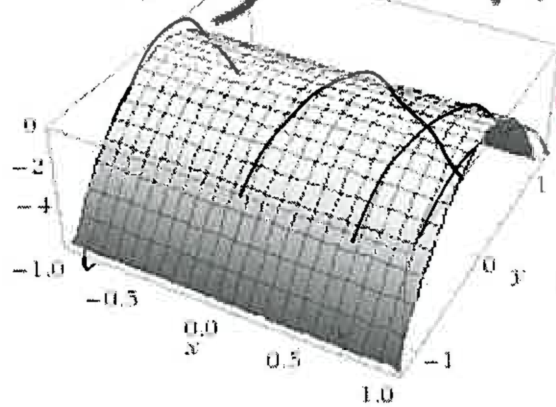



e.v = 0

e.v > 0

$Q(x,y) = -4y^2$ neg semi def

$$Q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

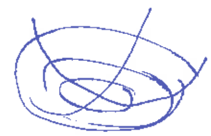



e.v = 0

e.v < 0

Defn and theorem:

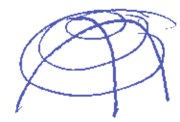
A symmetric matrix A is **positive definite**



if and only if the $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$

if and only if all the eigenvalues of A are positive. > 0

A symmetric matrix A is **negative definite**



if and only if the $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$

if and only if all the eigenvalues of A are negative. < 0

A symmetric matrix A is **indefinite**

saddle

if and only if the $\mathbf{x}^T A\mathbf{x}$ has both positive and negative values.

if and only if A ~~are~~ has positive and negative eigenvalues.

may or may not have a 0 e.value

A symmetric matrix A is **positive semidefinite**

if and only if the $\mathbf{x}^T A\mathbf{x} \geq 0$

if and only if all the eigenvalues of A are non-negative. ≥ 0

A symmetric matrix A is **negative semidefinite**

if and only if the $\mathbf{x}^T A\mathbf{x} \leq 0$

if and only if all the eigenvalues of A are non-positive. ≤ 0

negative definite $\Rightarrow$ negative semidefinite

< 0 $\qquad\qquad$ ≤ 0

Change of variable:

Let $\mathbf{x} = P\mathbf{y}$.

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T AP\mathbf{y} = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T \overbrace{(P^T AP)}^{D}\mathbf{y} = y^T D y$$

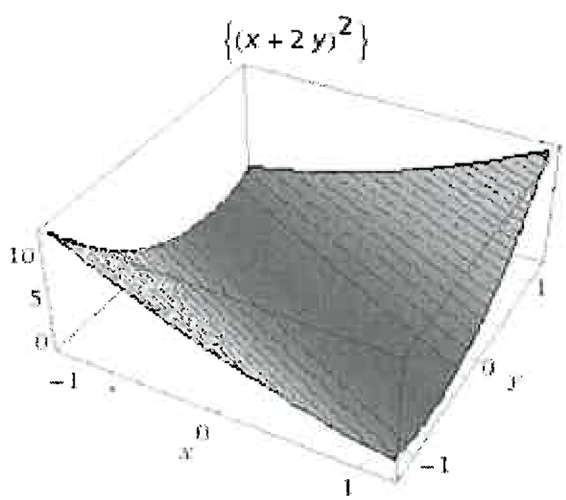
Suppose $A = PDP^{-1} = PDP^T$ where A is a symmetric matrix, D is diagonal, and P is orthonormal (i.e., $P^{-1} = P^T$).

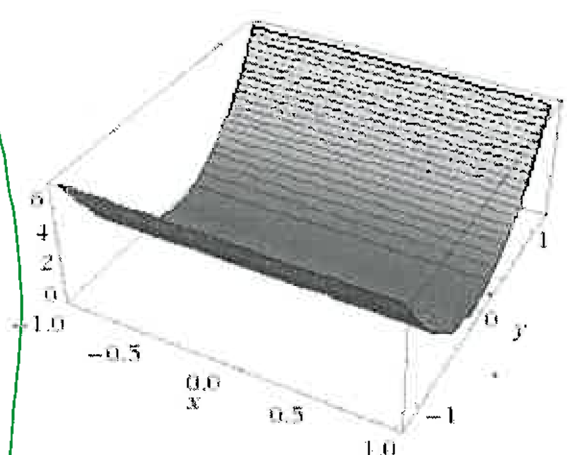
$A = PDP^T$ implies $P^T AP = P^T PDP^T P = D$

$Q(\mathbf{y}) = \mathbf{y}^T (P^T AP)\mathbf{y} = \mathbf{y}^T D\mathbf{y}$ diagonal

---

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$





$$Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}}_{D} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\lambda = 0, 5$

$A = PDP^T$

positive semi definite

Make a change of variable that transforms the following quadratic form into a quadratic form with no cross-product term:

$$Q(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Step 1: Orthogonally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

See section 7.1:

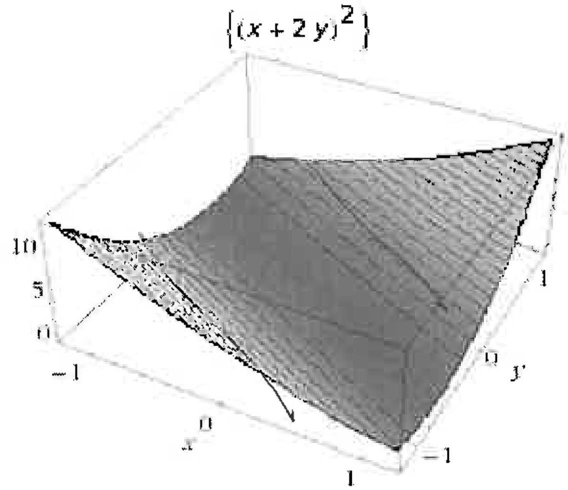
$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = A = PDP^T = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

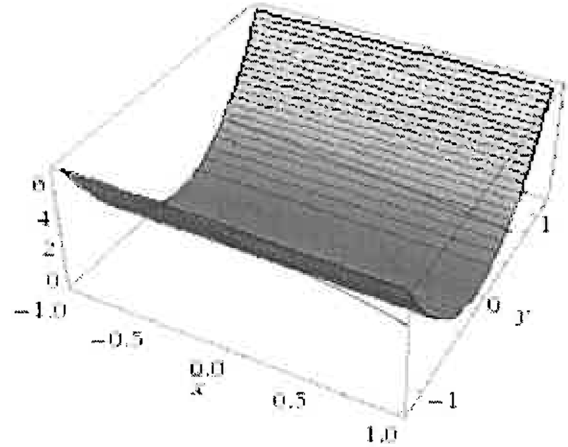
Step 2: Let $\mathbf{x} = P\mathbf{y}$

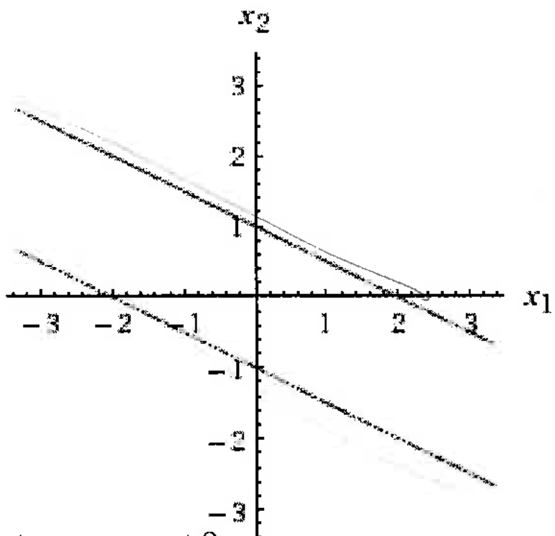
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{5}}y_1 + \frac{1}{\sqrt{5}}y_2 \\ \frac{1}{\sqrt{5}}y_1 + \frac{2}{\sqrt{5}}y_2 \end{bmatrix}$$

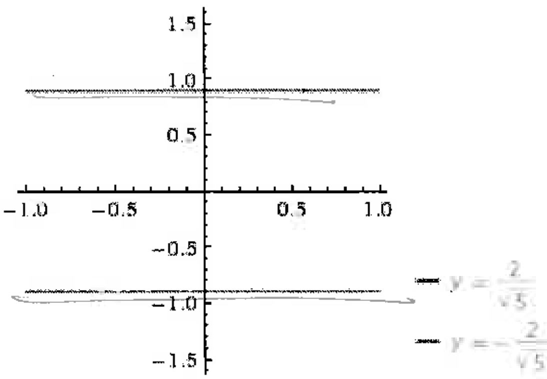
After change of variable:

$$Q(y_1, y_2) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$









$(x_1 + 2x_2)^2 = 4$

$((-\frac{2y_1}{\sqrt{5}} + \frac{y_2}{\sqrt{5}}) + 2(\frac{y_1}{\sqrt{5}} + \frac{2y_2}{\sqrt{5}}))^2 = 4$