$$\begin{bmatrix} 1 & 5 & 6 & 7 \\ 5 & 2 & 8 & 9 \\ 6 & 8 & 3 & 10 \\ 7 & 9 & 10 & 4 \end{bmatrix}$$

A symmetric if $A = A^T$

7.1: Special case $A = A^T$
combines Ch 5 & Ch 6

$A = A^T$ symmetry accross diagonal

Example 2:
Orthogonally diagonalize $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

Ch 5

$$det(A{-}\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -\lambda & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ -\lambda & -\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 1 \\ 1-\lambda & -1 \end{vmatrix}$$

$$= (1-\lambda)[(1-\lambda)(-\lambda) - \lambda] + [\lambda + \lambda]$$
$$= (1-\lambda)(-\lambda)[(1-\lambda) + 1] + 2\lambda = (1-\lambda)(-\lambda)(2-\lambda) + 2\lambda$$

Note I can factor out $-\lambda$, leaving only a quadratic to factor:

$$= -\lambda[(1-\lambda)(2-\lambda) - 2]$$
$$= -\lambda[\lambda^2 - 3\lambda + 2 - 2] = -\lambda[\lambda^2 - 3\lambda] = -\lambda^2[\lambda - 3]$$

$A - 0I$

Thus their are 2 eigenvalues:

if diag

2 fv

$\lambda = 0$ with algebraic multiplicity 2. Since A is symmetric, geometric multiplicity = algebraic multiplicity = 2. Thus the dimension of the eigenspace corresponding to $\lambda = 0$ $[= Nul(A - 0I) = Nul(A)]$ is 2.

$\Rightarrow$ A diag

$\lambda = 3$ w/algebraic multiplicity = 1 = geometric multiplicity.

1 fv

Thus we can find an orthogonal basis for R^3 where two of the basis vectors comes from the eigenspace corresponding to eigenvalue 0 while the third comes from the eigenspace corresponding to eigenvalue 3.

If not symmetric, check if diagonalizable by finding # of free variable = g.m.

Do you have enough l.i e. vectors to create square P (am = gm)

Solve $(A-0I)x=\vec{0}$

2.) Find a basis for each of the eigenspaces:

2a.) $\lambda = 0$: $A - 0I = A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

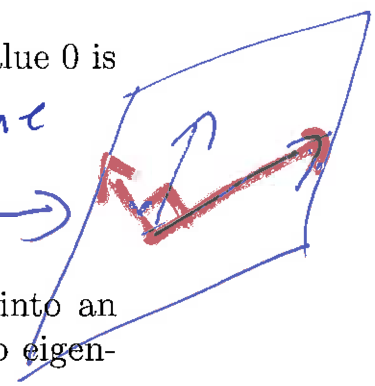
ch5

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2-d plane



since from same eigenvalue they don't have to be $\perp$

We can now use Gram-Schmidt to turn this basis into an orthogonal basis for the eigenspace corresponding to eigenvalue 0 or we can continue finding eigenvalues.

3a.) Create orthonormal basis using Gram-Schmidt for the eigenspace corresponding to eigenvalue 0:

ch 6

Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 1:

$$proj_{\mathbf{v_1}} \mathbf{v_2} = \left(\frac{\mathbf{v_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right) \mathbf{v_1} = \frac{-1+0+0}{1+1+0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

$\left(\frac{V_2 \cdot V_1}{V_1 \cdot V_1}\right) \vec{V_1}$

If A symmetric & asked to orthog diagonalize

Step 2: The vector component of $\mathbf{v_2}$ orthogonal to $\mathbf{v_1}$ is

$\perp$

$$\mathbf{v_2} - proj_{\mathbf{v_1}}\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Thus an orthogonal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}}_{v_2 - proj_{v_1} v_2} \right\}$$

Can check orthogonality via dot product = 0

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = 0$$

To create orthonormal basis, divide each vector by its length:

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\left\| \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$

$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \leftarrow \sqrt{1+1+2^2}$$

$$\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Thus an orthonormal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\}$$

unit length = 1

length = 1

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0/\sqrt{2} \end{bmatrix} \qquad \begin{bmatrix} -1/2 \,/\, \sqrt{3/2} \\ \frac{1}{2} \,/\, \sqrt{3/2} \\ 1 \,/\, \sqrt{3/2} \end{bmatrix} \qquad \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

Solve $(A-3I)\vec{x}=0$

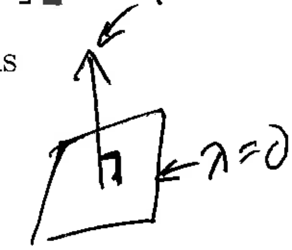
ch5

2b.) Find a basis for eigenspace corresponding to $\lambda = 3$:

$$A-3I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus a basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$



FYI: Alternate method to find 3rd vector: Since you have two linearly independent vectors from the eigenspace corresponding to eigenvalue 0, you only need one more vector which is orthogonal to these two to form a basis for R^3. Note since A is symmetric, any such vector will be an eigenvector of A with eigenvalue 3. Note this shortcut only works because we know what the eigenspace corresponding to eigenvalue 3 looks like: a line perpendicular to the plane representing the eigenspace corresponding to eigenvalue 0.

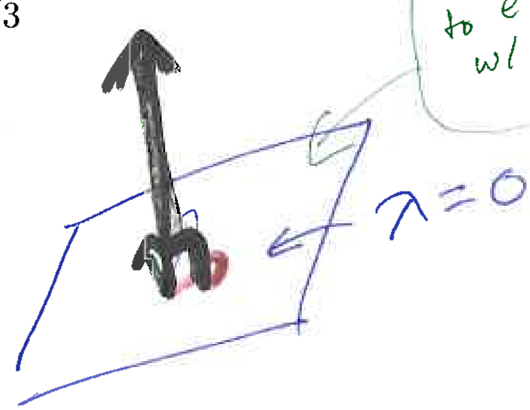
Ch 7:

A symmetric ⇒ e. vector w/ e. value 3 will be ⊥ to e. vector w/ e. value 0

3b.) Create orthonormal basis for the eigenspace corresponding to eigenvalue 3:

We only need to normalize:

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$



Thus orthonormal basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

ortho normal matrix

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

in 7.1 P^{-1} is easy to find

Thus $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Thus $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = A = PDP^{-1} =$

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}}_{P^{-1}}$$

Answer

Not true if A is NOT symmetric

7.1: Orthogonal Diagonalization

Equivalent Questions:

• Given an $n \times n$ matrix, does there exist an *orthonormal* basis for R^n consisting of *eigenvectors* of A?

• Given an $n \times n$ matrix, does there exist an orthonormal matrix P such that $P^{-1}AP = P^TAP$ is a diagonal matrix?

orthonormal

$PP^T = I$

• Is A symmetric?

Defn: A matrix is **symmetric** if $A = A^T$.

Recall An invertible matrix P is **ortho~~gonal~~** normal if $P^{-1} = P^T$

Defn: A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

Thm: If A is an $n \times n$ matrix, then the following are equivalent:

a.) A is orthogonally diagonalizable. $\Rightarrow$ normalize columns of P

b.) There exists an orthonormal basis for R^n consisting of eigenvectors of A.

|| orthonormal basis for $\mathbb{R}^n$

c.) A is symmetric.

Thm: If A is a symmetric matrix, then:

a.) The eigenvalues of A are all real numbers.

b.) Eigenvectors from different eigenspaces are orthogonal.

c.) Geometric multiplicity of an eigenvalue = its algebraic multiplicity

$A = A^T$

diagonalizable

7

Note

A symmetric $\Rightarrow$ A diag

" " $\Leftrightarrow$ A orthog diag

A orthog diag $\Rightarrow$ A symmetric

A diag $\Rightarrow$ A may or may not be symmetric

A is symmetric

$\Longrightarrow$ A diagonalizable

A is not symmetric

$\Rightarrow$ A may or may not be diagonalizable

(depends on if am = gm for all e.values

A diagonalizable $\Longrightarrow$ A may or may not be symmetric

---

A is orthogonally diagonalizable

$\Longleftrightarrow$ A is symmetric

can choose P to have orthogonal columns

7.1 cont

They don't need to be e. vectors to apply Gram-Schm just l. i.

Ch 6

Note if $\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ are linearly independent: Ch 6

(1.) You can use the Gram-Schmidt algorithm to find an orthogonal basis for $span\{\mathbf{v_1}, ..., \mathbf{v_n}\}$.

(2.) You can normalize these orthogonal vectors to create an orthonormal basis for $span\{\mathbf{v_1}, ..., \mathbf{v_n}\}$.

(3.) These basis vectors are not normally eigenvectors of $A = [\mathbf{v_1} ... \mathbf{v_n}]$ even if A is symmetric (note that there are an infinite number of orthogonal basis for $span\{\mathbf{v_1}, ..., \mathbf{v_n}\}$ even if $n = 2$ and $span\{\mathbf{v_1}, \mathbf{v_2}\}$is just a 2-dimensional plane)

Ch5

Note if A is a $n \times n$ square matrix that is diagonalizable, then you can find n linearly independent eigenvectors of A.

Each eigenvector is in $col(A)$: If $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, then $A\mathbf{v} = \lambda\mathbf{v}$. Thus $\frac{1}{\lambda}A\mathbf{v} = \mathbf{v}$. Hence $A(\frac{1}{\lambda}\mathbf{v}) = \mathbf{v}$. Thus $\mathbf{v}$ is in $col(A)$.

Thus $col(A)$ is an $n-$dimensional subspace of R^n. That is $col(A) = R^n$, and you can find a basis for $col(A) = R^n$ consisting of eigenvectors of A.

But these eigenvectors are NOT usually orthogonal UNLESS they come from different eigenspaces AND the matrix A is symmetric.

If A is NOT symmetric, then eigenvectors from different eigenspaces need NOT be orthogonal.

Multiple ways to find basis for $\mathbb{R}^4$

(1) Col(A) if A has no free variables and A is square

when you have enough li vectors

(2) Find e. vectors

usually different basis

Ch 6: Matrices may not be square

QR decomposition:

$$A = QR$$

Q is orthonormal
R is upper triangular

To find QR decomposition:

1.) Q: Use Gram-Schmidt to find orthonormal basis for column space of A

2.) Let $R = Q^T A$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1.) Use Gram-Schmidt to find orthonormal basis for column space of A

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1.) Use Gram-Schmidt to find orthonormal basis for column space of A

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1.) Use Gram-Schmidt to find orthogonal basis for column space of A

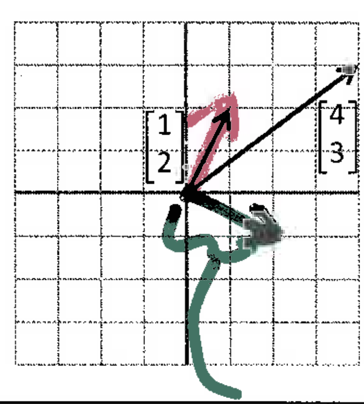
$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

1.) Use Gram-Schmidt to find orthogonal basis for column space of A

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, ? \right\}$$

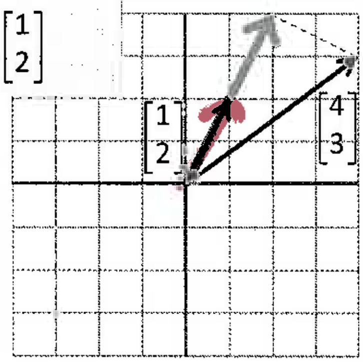


?

For QR
Keep first vector

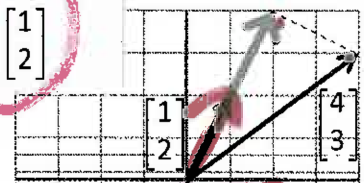
Find orthogonal projection

of $\begin{bmatrix}4\\3\end{bmatrix}$ onto $\begin{bmatrix}1\\2\end{bmatrix}$



Find orthogonal projection

of $\begin{bmatrix}4\\3\end{bmatrix}$ onto $\begin{bmatrix}1\\2\end{bmatrix}$

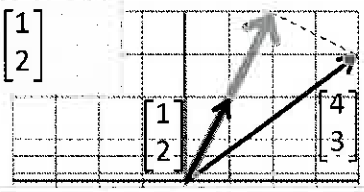


$$\text{proj}_{\begin{bmatrix}1\\2\end{bmatrix}}\begin{bmatrix}4\\3\end{bmatrix} = \frac{(1,2)\bullet(4,3)}{(1,2)\bullet(1,2)}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{1(4)+2(3)}{1^2+2^2}\begin{bmatrix}1\\2\end{bmatrix} = \frac{4+6}{1+4}\begin{bmatrix}1\\2\end{bmatrix}$$

Find orthogonal projection

of $\begin{bmatrix}4\\3\end{bmatrix}$ onto $\begin{bmatrix}1\\2\end{bmatrix}$

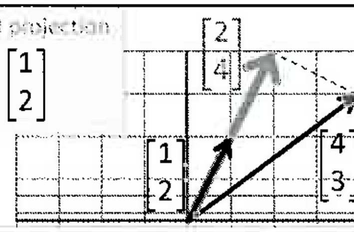


$$\text{proj}_{\begin{bmatrix}1\\2\end{bmatrix}}\begin{bmatrix}4\\3\end{bmatrix} = \frac{(1,2)\bullet(4,3)}{(1,2)\bullet(1,2)}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{1(4)+2(3)}{1^2+2^2}\begin{bmatrix}1\\2\end{bmatrix} = \frac{10}{5}\begin{bmatrix}1\\2\end{bmatrix}$$

Find orthogonal projection

of $\begin{bmatrix}4\\3\end{bmatrix}$ onto $\begin{bmatrix}1\\2\end{bmatrix}$

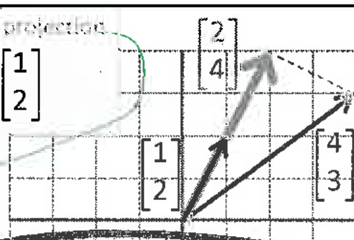


$$\text{proj}_{\begin{bmatrix}1\\2\end{bmatrix}}\begin{bmatrix}4\\3\end{bmatrix} = \frac{(1,2)\bullet(4,3)}{(1,2)\bullet(1,2)}\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \frac{1(4)+2(3)}{1^2+2^2}\begin{bmatrix}1\\2\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

Find orthogonal projection

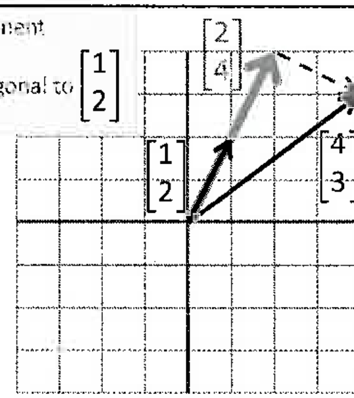
of $\begin{bmatrix}4\\3\end{bmatrix}$ onto $\begin{bmatrix}1\\2\end{bmatrix}$



$$\text{proj}_{\begin{bmatrix}1\\2\end{bmatrix}}\begin{bmatrix}4\\3\end{bmatrix} = \frac{(1,2)\bullet(4,3)}{(1,2)\bullet(1,2)}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

Find the component

of $\begin{bmatrix}4\\3\end{bmatrix}$ orthogonal to $\begin{bmatrix}1\\2\end{bmatrix}$



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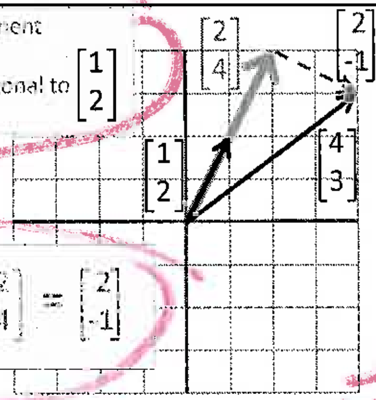
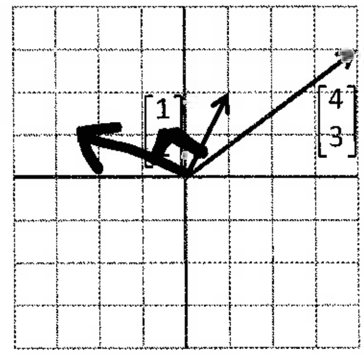
Find the component of $\begin{bmatrix}4\\3\end{bmatrix}$ orthogonal to $\begin{bmatrix}1\\2\end{bmatrix}$



$$\begin{bmatrix}4\\3\end{bmatrix} - \begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\vec{v}_2 - \text{proj}_{\vec{v}_1} \vec{v}_2$$

Short-cut for R^2 case:



Find the QR decomposition of

$$A = \begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}$$

1.) Use Gram-Schmidt to find orthogonal basis for column space of A

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\right\}$$

Find the length of each vector:

$$\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\left\|\begin{bmatrix}2\\-1\end{bmatrix}\right\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Divide each vector by its length:

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}\right\}$$

$$= \text{span}\left\{\begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}\frac{2}{\sqrt{5}}\\ -\frac{1}{\sqrt{5}}\end{bmatrix}\right\}$$

o.n basis for colA

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}\frac{2}{\sqrt{5}}\\ -\frac{1}{\sqrt{5}}\end{bmatrix}\right\}$$

$$Q = \begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\end{bmatrix}$$

⊥ perpend unit vec

A = QR

$$Q^{-1} = Q^T$$

since Q orthonormal

$$Q^T A = \cancel{Q^T Q} R$$

$$Q^T A = R$$

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$$A = QR$$

$$A = QR$$

$$Q^{-1}A = Q^{-1}QR$$

$$Q^{-1}A = R$$

Q has orthonormal columns:

Thus $Q^{-1} = Q^T$

Thus $R = Q^{-1}A = Q^TA$

Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = QR$$

$$R = Q^{-1}A = Q^TA = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{\sqrt{5}} & \frac{10}{\sqrt{5}} \\ 0 & \frac{5}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \sqrt{5} & 2\sqrt{5} \\ 0 & \sqrt{5} \end{bmatrix}$$

Note R is upper triangular

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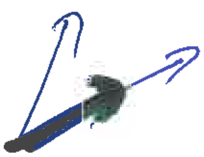
Thm: Let $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal basis for an inner product space V. Let $\mathbf{a}$ be an arbitrary vector in V. Then

$$\mathbf{a} = c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}$$

$$\text{where } c_j = \frac{<\mathbf{a}, \mathbf{v_j}>}{||\mathbf{v_j}||^2} \text{ for } j = 1, 2, ..., n.$$

Note if $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is an orthonormal basis, then

$$||\mathbf{v_j}|| = 1 \text{ and } c_j =< \mathbf{a}, \mathbf{v_j} >$$



Thm: Let $\mathbf{a}$, $\mathbf{v}$ be nonzero vectors in R^k.

The vector component of $\mathbf{a}$ along $\mathbf{v}$

$$= \text{orthogonal projection of } \mathbf{a} \text{ on } \mathbf{v}$$

$$= proj_{\mathbf{v}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$$

The vector component of $\mathbf{a}$ orthogonal to $\mathbf{v}$

$$= \mathbf{a} - proj_{\mathbf{v}}\mathbf{a} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{v}}{||\mathbf{v}||^2}\mathbf{v}$$

Thm: Let $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal basis for subspace W of an inner product space V. Let $\mathbf{a}$ be an arbitrary vector in V. Then

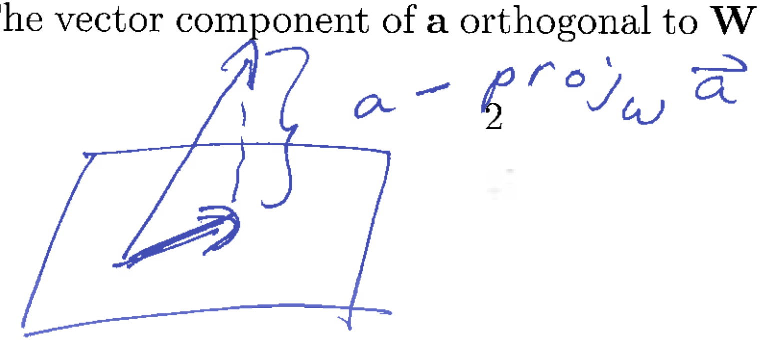
$$proj_W\mathbf{a} = c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}$$

$$\text{where } c_j = \frac{<\mathbf{a}, \mathbf{v_j}>}{||\mathbf{v_j}||^2} \text{ for } j = 1, 2, ..., n.$$

Note if $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is an orthonormal basis, then

$$||\mathbf{v_j}|| = 1 \text{ and } c_j =< \mathbf{a}, \mathbf{v_j} >$$

The vector component of $\mathbf{a}$ orthogonal to $\mathbf{W}$ = $\mathbf{a} - proj_{\mathbf{W}}\mathbf{a}$



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Thm (Gram-Schmidt process for constructing an orthogonal basis):

Let $\mathcal{T} = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ be a basis for an inner product space V. Let $\mathcal{T}' = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be defined as follows:

$$\mathbf{v_1} = \mathbf{a_1}$$

$$\mathbf{v_2} = \mathbf{a_2} - \frac{<\mathbf{a_2},\mathbf{v_1}>}{<\mathbf{v_1},\mathbf{v_1}>}\mathbf{v_1}$$

$$\mathbf{v_3} = \mathbf{a_3} - \frac{<\mathbf{a_3},\mathbf{v_1}>}{<\mathbf{v_1},\mathbf{v_1}>}\mathbf{v_1} - \frac{<\mathbf{a_3},\mathbf{v_2}>}{<\mathbf{v_2},\mathbf{v_2}>}\mathbf{v_2}$$

$$\vdots$$

$$\mathbf{v_n} = \mathbf{a_n} - \frac{<\mathbf{a_n},\mathbf{v_1}>}{<\mathbf{v_1},\mathbf{v_1}>}\mathbf{v_1} - \frac{<\mathbf{a_n},\mathbf{v_2}>}{<\mathbf{v_2},\mathbf{v_2}>}\mathbf{v_2} - ... - \frac{<\mathbf{a_n},\mathbf{v_n}>}{<\mathbf{v_n},\mathbf{v_n}>}\mathbf{v_n}$$

Then the set $\mathcal{T}'$ is an orthogonal basis for V.

An orthonormal basis for V is given by

$$\mathcal{T}'' = \left\{ \frac{\mathbf{v_1}}{||\mathbf{v_1}||}, \frac{\mathbf{v_2}}{||\mathbf{v_2}||}, ..., \frac{\mathbf{v_n}}{||\mathbf{v_n}||} \right\}.$$

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