11/19

7.1

Special case $A = A^T$

If $\vec{v}_1$ is e.vector of A w/ e.value λ_1

$\vec{v}_2$ " " " λ_2

If $\lambda_1 \neq \lambda_2 \Rightarrow$ (If $A = A^T$, then $\vec{v}_1 \perp V_2$)

Suppose $A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $A\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

Claim:

If $\boxed{A = A^T}$ and $\lambda_1 \neq \lambda_2$, then $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

I.e., If eigenvectors come from different eigenspaces, then the eigenvectors are orthogonal WHEN $\boxed{A = A^T}$.

Proof

Pf of claim: $\lambda_1(v_1, v_2) \cdot (w_1, w_2) = \lambda_1[v_1, v_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

trick

$$= (\lambda_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix})^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = (A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix})^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T A^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \underset{\text{hypothesis}}{=} [v_1, v_2]A\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= [v_1, v_2]\lambda_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_2[v_1, v_2]\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$= \lambda_2(v_1, v_2) \cdot (w_1, w_2)$$

$\lambda_1(v_1, v_2) \cdot (w_1, w_2) = \lambda_2(v_1, v_2) \cdot (w_1, w_2)$

implies $\lambda_1(v_1, v_2) \cdot (w_1, w_2) - \lambda_2(v_1, v_2) \cdot (w_1, w_2) = 0$.

Thus $(\lambda_1 - \lambda_2)[(v_1, v_2) \cdot (w_1, w_2)] = 0$

$\lambda_1 \neq \lambda_2$ implies $(v_1, v_2) \cdot (w_1, w_2) = 0$

Thus these eigenvectors are orthogonal.

7.1: Orthogonal Diagonalization

Equivalent Questions:

• Given an $n \times n$ matrix, does there exist an *orthonormal* basis for R^n consisting of *eigenvectors* of A?

• Given an $n \times n$ matrix, does there exist an orthonormal matrix P such that $P^{-1}AP = P^TAP$ is a diagonal matrix?

• Is A symmetric?

orthonormal $PP^T = I$

Defn: A matrix is **symmetric** if $\boxed{A = A^T}$.

Recall An invertible matrix P is ~~**orthogonal**~~ normal if $P^{-1} = P^T$

Defn: A matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

Thm: If A is an $n \times n$ matrix, then the following are equivalent:

a.) A is orthogonally diagonalizable. ⇒ normalize columns of P

b.) There exists an orthonormal basis for R^n consisting of eigenvectors of A.

|| orthonormal basis for R^n

c.) A is symmetric.

Thm: If A is a $\boxed{\text{symmetric}}$ matrix, then:

a.) The eigenvalues of A are all real numbers.

b.) Eigenvectors from different eigenspaces are orthogonal.

c.) Geometric multiplicity of an eigenvalue = its algebraic multiplicity

diagonalizable

$A = A^T$

IF A is symmetric,

To orthogonally diagonalize a symmetric matrix A:

1.) Find the eigenvalues of A.

$$\text{Solve } det(A - \lambda I) = 0 \text{ for } \lambda.$$

2.) Find a basis for each of the eigenspaces.

$$\text{Solve } (A - \lambda_j I)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

3.) Use the Gram-Schmidt process to find an orthonormal basis for each eigenspace.

That is for each λ_j use Gram-Schmidt to find an orthonormal basis for $Nul(A - \lambda_j I)$.

Eigenvectors from different eigenspaces will be orthogonal, so you don't need to apply Gram-Schmidt to eigenvectors from different eigenspaces

4.) Use the eigenvalues of A to construct the diagonal matrix D, and use the orthonormal basis of the corresponding eigenspaces for the corresponding columns of P.

5.) Note $P^{-1} = P^T$ since the columns of P are orthonormal.

ch 6

$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = A$

Example 1:

Orthogonally diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

ch 5

Step 1: Find the eigenvalues of A:

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 4 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

Thus $\lambda = 0, 5$ are are eigenvalues of A.

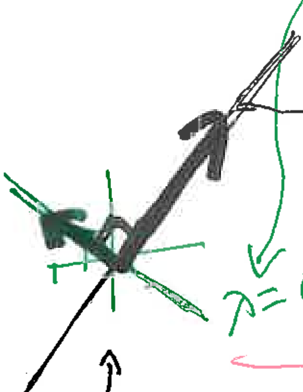
2.) Find a basis for each of the eigenspaces:

ch 5

$\lambda = 0$: $(A - 0I) = A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

Thus $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 0.

$\lambda = 5$: $(A - 5I) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$



$\lambda = 0$

Thus $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A with eigenvalue 5.

The two e:space must be $\perp$ to each other

3.) Create orthonormal basis:

ch 6

Since A is symmetric and the eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ come from different eigenspaces (ie their eigenvalues are different), these eigenvectors are orthogonal. Thus we only

7.1: $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -2 + 2 = 0$

Ch 6

need to normalize them:

$$\left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| = \sqrt{4+1} = \sqrt{5}$$

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1+4} = \sqrt{5}$$

Thus an orthonormal basis for $col(A) = R^2 = \left\{ \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\}$ $\blacksquare$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 \\ 0 & \boxed{5} \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \boxed{\frac{1}{\sqrt{5}}} \\ \frac{1}{\sqrt{5}} & \boxed{\frac{2}{\sqrt{5}}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $\underline{P^{-1} = P^T}$

Thus $P^{-1} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$

Note that in this example, $P^{-1} = P$, but that is NOT normally the case.

Thus $A = PDP^{-1}$

Thus $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}}_{P^{-1}}$

$A = A^T$ symmetry accross diagonal

Example 2:
Orthogonally diagonalize $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Step 1: Find the eigenvalues of A:

$$det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -\lambda & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 1 \\ -\lambda & -\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 1 \\ 1-\lambda & -1 \end{vmatrix}$$

$= (1-\lambda)[(1-\lambda)(-\lambda) - \lambda] + [\lambda + \lambda]$
$= (1-\lambda)(-\lambda)[(1-\lambda)+1] + 2\lambda = (1-\lambda)(-\lambda)(2-\lambda) + 2\lambda$
Note I can factor out $-\lambda$, leaving only a quadratic to factor:
$= -\lambda[(1-\lambda)(2-\lambda) - 2]$
$= -\lambda[\lambda^2 - 3\lambda + 2 - 2] = -\lambda[\lambda^2 - 3\lambda] = -\lambda^2[\lambda - 3]$

Thus their are 2 eigenvalues:

$\lambda = 0$ with algebraic multiplicity 2. Since A is symmetric, geometric multiplicity = algebraic multiplicity = 2. Thus the dimension of the eigenspace corresponding to $\lambda = 0$
$[= Nul(A - 0I) = Nul(A)]$ is 2.

$\lambda = 3$ w/algebraic multiplicity = 1 = geometric multiplicity.

Thus we can find an orthogonal basis for R^3 where two of the basis vectors comes from the eigenspace corresponding to eigenvalue 0 while the third comes from the eigenspace corresponding to eigenvalue 3.

EX Orthonormally diagonalize

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

① DIAGONALIZE (Ch5)

1a) Find e. value

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1-1 & -1+1-\lambda & 1-\lambda-1 \end{vmatrix}$$

$$\xrightarrow[R_2+R_3 \to R_3]{} \begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -\lambda \end{vmatrix} =$$

$$= +0 - (-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & -1 \end{vmatrix} + (-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}$$

$$\lambda\Big((1-\lambda)(-1)-(-1)(1)\Big)+(-\lambda)\Big((1-\lambda)^2-(-1)(-1)\Big)$$

$$=\lambda\left[\left(\cancel{-1}+\lambda+\cancel{1}\right)-\left(\cancel{1}-2\lambda+\lambda^2-\cancel{1}\right)\right]$$

$$=\lambda\left[3\lambda-\lambda^2\right]$$

$$=-\lambda^2\left[\lambda-3\right]=0$$

$\Rightarrow$ $\lambda=0$ | $\lambda=3$

alg mult $=2$ | alg mult $=1$

$\parallel$ | $\parallel$

since $A=A^T$ geom mult | g. m.

2 free variables | 1 free variables

Ch 5 Find basis for each e. space

$$\boxed{\text{Solve } (A - 0I)\vec{x} = \vec{0}}$$

a.m = g.m = 2 = # of free variables

$\lambda = 0$: $A - 0I =$

$$= \begin{bmatrix} 1 & -1 & 1 \\ -1 \; {+1} & 1 \; {-1} & -1 \; {+1} \\ 1 \; {-1} & -1 \; {+1} & 1 \; {-1} \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

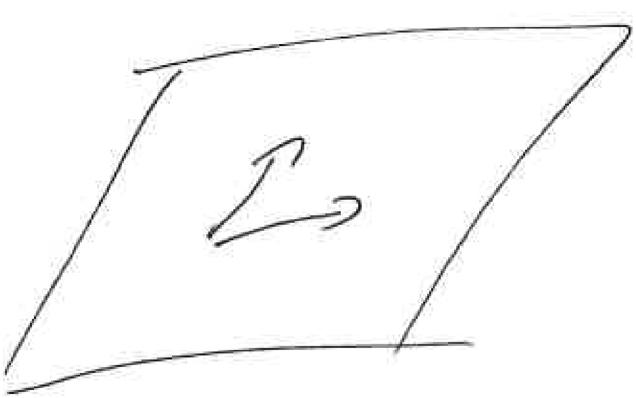
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$$

E. space for $\lambda = 0$

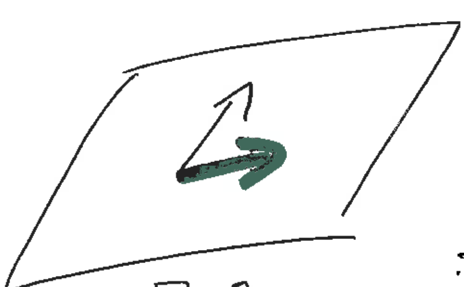
$$= \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

At some point, we want our basis to be orthonormal

$$\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\underline{\underline{\text{NOT}}}\ \text{ORTHOGONAL}} = -1+0+0 = -1 \neq 0$$



We will use GRAM-SCHMIDT TO CREATE ORTHOGONAL BASIS & then normalize

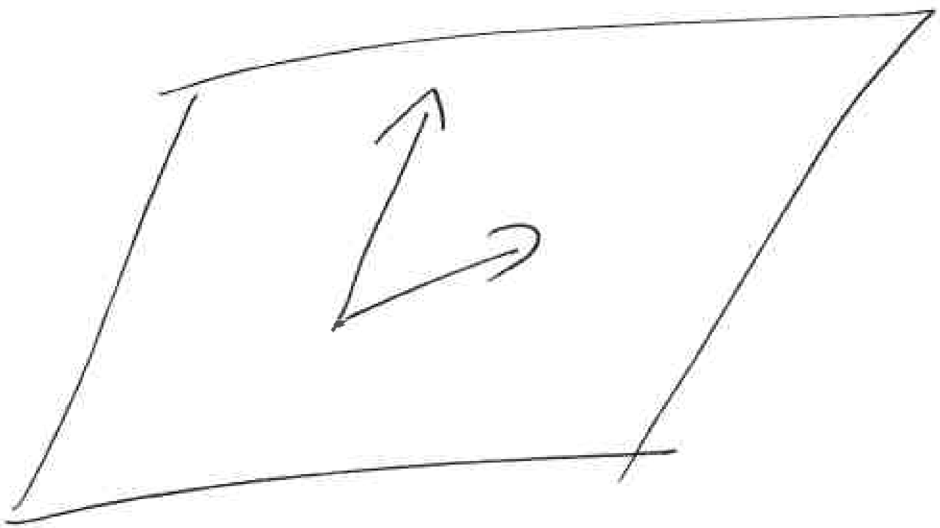


$$\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, ? \right\}$$

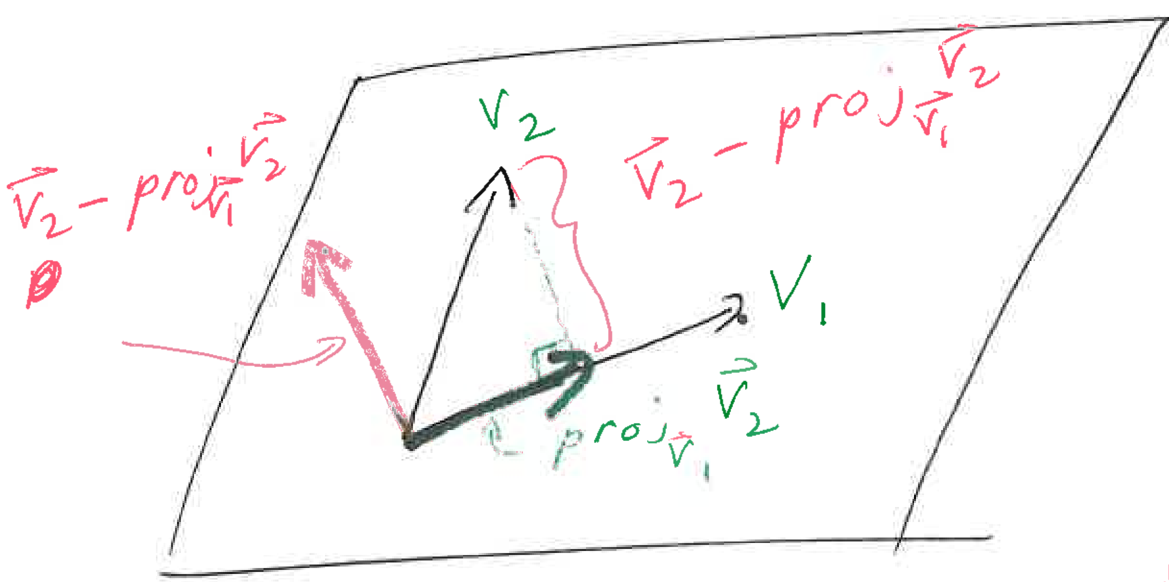
Ch 6

E space fo $\lambda = 0$

$\boxed{\text{Ch6}}$



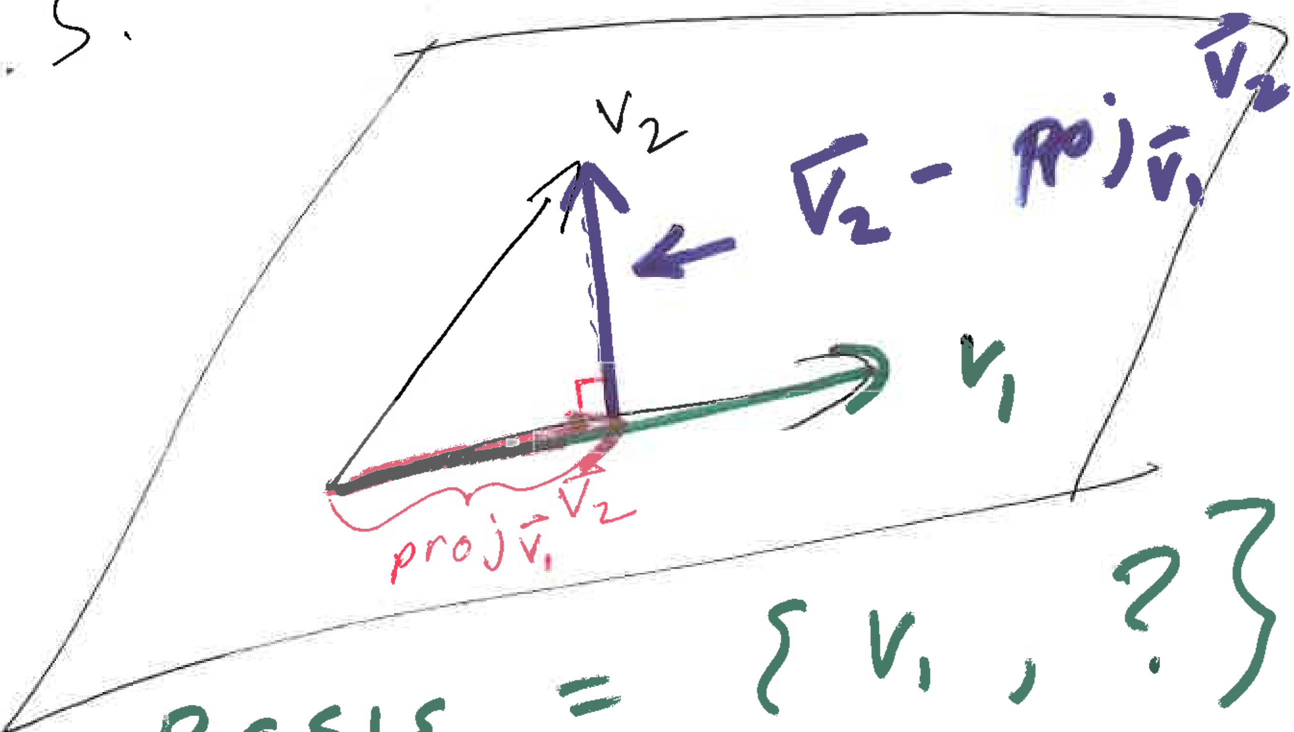
At some point use Gram-Schmidt to find orthogonal basis and then normalize



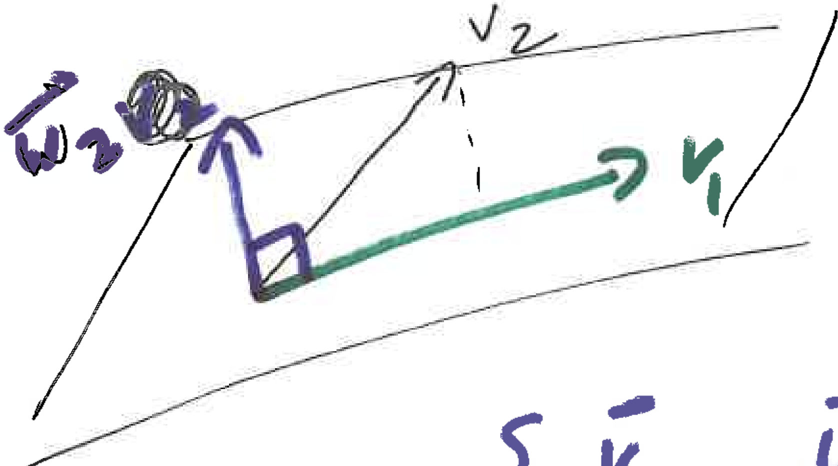
$\vec{v}_2 - \text{proj}_{\vec{v}_1} \vec{v}_2$

$\text{span}\{v_1, v_2\} = \text{span}\{v_1, ?\}$

G.S.



$$\text{Basis} = \{ v_1 \,,\ ? \}$$



$$\text{Basis} = \{ \vec{v}_1 \,, \vec{w}_2 \}$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{v}_1} \vec{v}_2$$

$$\text{span} \{ \vec{v}_1 , \vec{v}_2 \} = \text{span} \{ \vec{v}_1 , \vec{w} \}$$

espace for $\lambda = 0$:

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

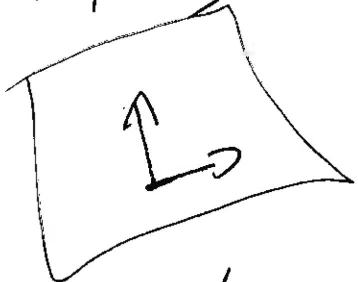
$$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, ? \right\}$$

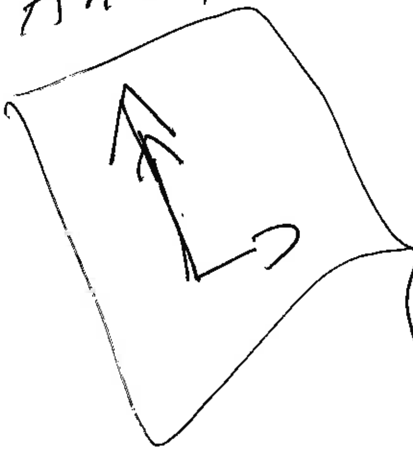
$$\operatorname{proj}_{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{(1,1,0)\cdot(-1,0,1)}{(1,1,0)\cdot(1,1,0)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{-1+0+0}{1+1+0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ \cancel{0}\,1 \end{bmatrix}$$

orthog basis  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$

Another orthog basis  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$

Normalize

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$$

$$\left\| \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\| = \sqrt{(-1)^2+(1)^2+2^2} = \sqrt{6}$$

orthonormal basis is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$$

e. space $\lambda = 0$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{Span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}}_{\text{orthog basis}}$$

~~orth~~

$\times(2)$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$\downarrow$

$$\text{span}\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$$

$\lambda = 3$: $A - 3I$ e. space
Nulspace$(A-3I)$

$$\begin{bmatrix} 1-3 & -1 & 1 \\ -1 & 1-3 & -1 \\ 1 & -1 & 1-3 \end{bmatrix} = \begin{array}{ccc} +2 & -2 & -4 \\ \end{array}$$
$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \rightarrow \begin{array}{ccc} & & -1 \end{array}$$
$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3$$

Ch 6: Normalize

$$\left\|\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$$

Normalize basis for e. space for $\lambda = 3$

$$\begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$$

$$D = \begin{bmatrix}0 & & \\ & 0 & \\ & & 3\end{bmatrix} \quad P = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 3 \end{bmatrix} \qquad P = \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}}_{\text{o.n.}}$$

2.) Find a basis for each of the eigenspaces:

2a.) $\lambda = 0 : A - 0I = A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Thus a basis for eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

We can now use Gram-Schmidt to turn this basis into an orthogonal basis for the eigenspace corresponding to eigenvalue 0 or we can continue finding eigenvalues.

3a.) Create orthonormal basis using Gram-Schmidt for the eigenspace corresponding to eigenvalue 0:

Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$proj_{\mathbf{v_1}} \mathbf{v_2} = \frac{\mathbf{v_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} = \frac{-1+0+0}{1+1+0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

The vector component of $\mathbf{v_2}$ orthogonal to $\mathbf{v_1}$ is

$$\mathbf{v_2} - proj_{\mathbf{v_1}}\mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Thus an orthogonal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

To create orthonormal basis, divide each vector by its length:

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

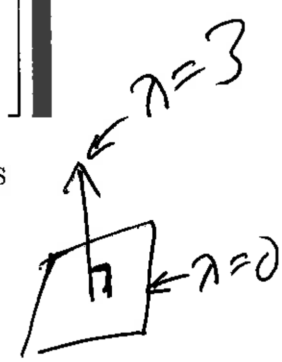
$$\left\| \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$

Thus an orthonormal basis for the eigenspace corresponding to eigenvalue 0 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix} \right\}$$

2b.) Find a basis for eigenspace corresponding to $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



Thus a basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

FYI: Alternate method to find 3rd vector: Since you have two linearly independent vectors from the eigenspace corresponding to eigenvalue 0, you only need one more vector which is orthogonal to these two to form a basis for R^3. Note since A is symmetric, any such vector will be an eigenvector of A with eigenvalue 3. Note this shortcut only works because we know what the eigenspace corresponding to eigenvalue 3 looks like: a line perpendicular to the plane representing the eigenspace corresponding to eigenvalue 0.

3b.) Create orthonormal basis for the eigenspace corresponding to eigenvalue 3:

We only need to normalize:

$$\left\| \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$$

Thus orthonormal basis for eigenspace corresponding to eigenvalue 3 is

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

4.) Construct D and P

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Make sure order of eigenvectors in D match order of eigenvalues in P.

5.) P orthonormal implies $P^{-1} = P^T$

Thus $P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Thus $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = A = PDP^{-1} =$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$