11/12

6.1: Inner Products.

Defn: Let V be a vector space over the real numbers. An **inner product** for V is a function that associates a real number $\mathbf{u} \cdot \mathbf{v}$ to every pair of vectors, $\mathbf{u}$ and $\mathbf{v}$ in V such that the following properties are satisfied for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and scalars c:

pythogorean thm

a.) $\underline{\mathbf{u} \cdot \mathbf{v}} = \underline{\mathbf{v} \cdot \mathbf{u}}$

b.) $\underline{\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

$$\sum_{i=1}^{n} u_i^2 = \text{length}^2$$

c.) $(c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d.) $\boxed{\mathbf{u} \cdot \mathbf{u} \geq 0 \text{ and } \mathbf{u} \cdot \mathbf{u} = 0 \text{ if and only if } \mathbf{u} = \mathbf{0}}$

A vector space V together with an inner product is called an **inner product space.**

Thm 6.1.1′: Let V be an inner product space. Then for all vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{v}$ in V and scalars c_1, c_2:

a.) $(c_1\mathbf{u_1} + c_2\mathbf{u_2}) \cdot \mathbf{v} = \mathbf{v} \cdot (c_1\mathbf{u_1} + c_2\mathbf{u_2})$
$= c_1(\mathbf{u_1} \cdot \mathbf{v}) + c_2(\mathbf{u_2} \cdot \mathbf{v})$

b.) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

Inner Product Example: Dot product on R^n.

Defn: $\Sigma_{k=1}^{m} a_k = a_1 + a_2 + \ldots + a_m$

Defn:
The **dot product** of $\mathbf{u} = (u_1, ..., u_m)$ & $\mathbf{v} = (v_1, ..., v_m)$ is

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \Sigma_{k=1}^{m} u_k v_k.}$$

In words, $\mathbf{u} \cdot \mathbf{v}$ is the sum of the products of the corresponding components of $\mathbf{u}$ and $\mathbf{v}$.

Note that $\boxed{\mathbf{u} \cdot \mathbf{v} \text{ is a real number (not a vector).}}$

Examples:

$(1, 2, 3) \cdot (4, 5, 6) = (1)(4) + (2)(5) + 3(6)$

$= 4 + 10 + 18$

$= 32$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = (1)(-2) + (2)(1)$

$= -2 + 2 = 0$

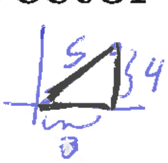
~~its~~

Defn: Let $\mathbf{v}$ be a vector in an inner product space $\mathbf{V}$. The **length** or **norm** of $\mathbf{v} = ||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >}$. $= \sqrt{V \cdot V}$

$||(3,4)|| = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$

Defn: The vector $\mathbf{u}$ is a **unit vector** if $||\mathbf{u}|| = 1.$

pythagorean theorem



Note that $\boxed{\frac{\mathbf{v}}{||\mathbf{v}||}}$ is a unit vector.

Create a unit vector in the direction of the vector (3, 4):

$$||(3,4)|| = \sqrt{9+16} = \sqrt{25} = 5$$

$$\boxed{\left(\frac{3}{5}, \frac{4}{5}\right)}$$

Create a unit vector in the direction of the vector (1, 2):

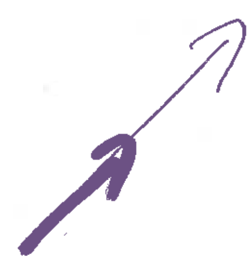
$$||(1,2)|| = \sqrt{1^2+2^2} = \sqrt{5}$$

$$\boxed{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)}$$

Create a unit vector in the direction of the vector (-2, 1):

$$||(-2,1)|| = \sqrt{(-2)^2+1^2} = \sqrt{5}$$

$$\boxed{\left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)}$$



Defn: **u** and **v** are **orthogonal** (or **perpendicular**) if

$$\boxed{\mathbf{u} \cdot \mathbf{v} = 0.}$$

Example: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1(-2) + 2(1) = 0$

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a set of orthogonal ~~unit~~ vectors.

Example: $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{-2}{\sqrt{5}}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right) = \frac{-2+2}{5}$

$= 0$

Thus $\left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

Observation: $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} + \frac{4}{5} & \frac{-2}{5} + \frac{2}{5} \\ \frac{-2}{5} + \frac{2}{5} & \frac{4}{5} + \frac{1}{5} \end{bmatrix}$

Suppose $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is a pair of orthogonal unit vectors. Then

$$\underset{A}{\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}} = \begin{bmatrix} \mathbf{u} \cdot \mathbf{u} & \mathbf{u} \cdot \mathbf{v} \\ \mathbf{v} \cdot \mathbf{u} & \mathbf{v} \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If rows (or columns) of A are orthogonal unit vector

$$\boxed{A^{-1} = A^T}$$

$$A A^T = I$$

$$= \begin{bmatrix} \overset{1}{u_1 \cdot u_1} & \overset{0}{u_1 \cdot u_2} & \\ & \overset{1}{u_2 \cdot u_2} & \\ & & \ddots \end{bmatrix}$$

Orthonormal Bases.

A set of vectors, $\mathcal{S}$, is an **orthogonal set** if every pair of distinct vectors is orthogonal.

A set $\mathcal{T}$, is an **orthonormal set** if it is an orthogonal set and if every vector in $\mathcal{T}$ has norm equal to 1.

& unique soln $\Leftarrow$ l indep

Thm: Let $\mathcal{T} = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal set of nonzero vectors in an inner product space V. Then $\mathcal{T}$ is linearly independent.

$A = [v_1 \ldots v_n]$. Solve $A\vec{x} = \vec{0}$

$[v_1 \ldots v_n]\vec{x} = 0$

Cor: An orthonormal set of vectors is linearly independent.

$$\vec{V}_1 \cdot (\vec{V}_1 x_1 + \vec{V}_2 x_2 + \ldots + \vec{V}_n x_n) = 0 \cdot \vec{V}_1$$

$$(\vec{V}_1 \cdot \vec{V}_1)x_1 + \underbrace{(\vec{V}_1 \cdot \vec{V}_2)}_{0}x_2 + \ldots + \underbrace{(\vec{V}_1 \cdot \vec{V}_n)}_{0}x_n = 0$$

$$\Rightarrow (\vec{V}_1 \cdot \vec{V}_1)x_1 = 0 \Rightarrow x_1 = \frac{0}{\vec{V}_1 \cdot \vec{V}_1} = 0$$

since $\vec{V}_1 \neq 0$ $\Rightarrow \vec{V}_1 \cdot \vec{V}_1 \neq 0$

Defn: Let $\mathbf{V}$ be an inner product space. If $\boxed{V = spanT}$ &

i.) if $\mathcal{T}$ is an orthogonal set, then $\mathcal{T}$ is an **orthogonal basis** for V. consisting of nonzero vectors

l indep

ii.) if $\mathcal{T}$ is orthonormal set, then $\mathcal{T}$ is an **orthonormal basis** for V.

Similarly $x_i = 0$ for all i

$\Rightarrow$ unique soln

$\Rightarrow$ columns are l.i

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

columns are orthog $\Rightarrow$ l. indep $\Leftrightarrow$ unique soln

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 \right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}} x_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$5 x_1 + 0 x_2 = 13$$

$$5 x_1 = 13$$

$$\boxed{\begin{array}{l} [v_1 \ v_2] \vec{x} = \vec{b} \\ v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ b = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \end{array}}$$

$$x_1 = \frac{13}{5} = \frac{\vec{v}_1 \cdot \vec{b}}{\vec{v}_1 \cdot \vec{v}_1}$$

$$\vec{v}_2 \cdot (\vec{v}_1 x_1 + \vec{v}_2 x_2) = \vec{b} \cdot \vec{v}_2$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 \right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$(\vec{v}_2 \cdot \vec{v}_1) x_1 + (\vec{v}_2 \cdot \vec{v}_2) x_2 = \vec{b} \cdot \vec{v}_2$$

$$0 x_1 + 5 x_2 = -1$$

$$\Rightarrow x_2 = -\frac{1}{5} = \frac{\vec{b} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}$$

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orthogonal set

If $\vec{b}$ is in span $\{v_1 \dots v_n\}$

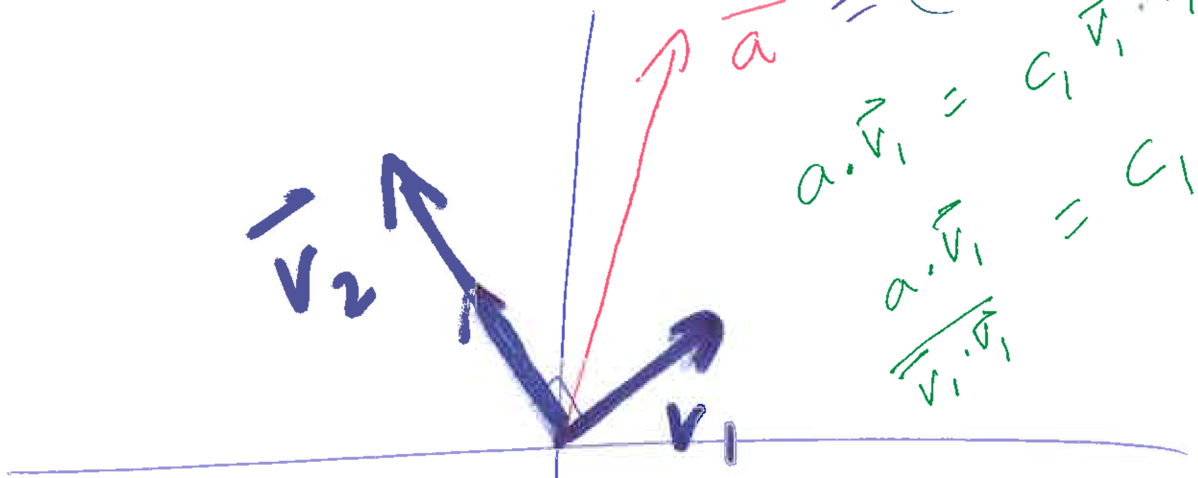
$$\vec{b} = \sum_{i=1}^{n} c_i v_i$$

where $c_i = \dfrac{\vec{b} \cdot \vec{v_i}}{\vec{v_i} \cdot \vec{v_i}}$

$$\vec{a}^{\,v_1} = (c_1 \vec{v}_1 + c_2 \vec{v}_2)\cdot v_1$$

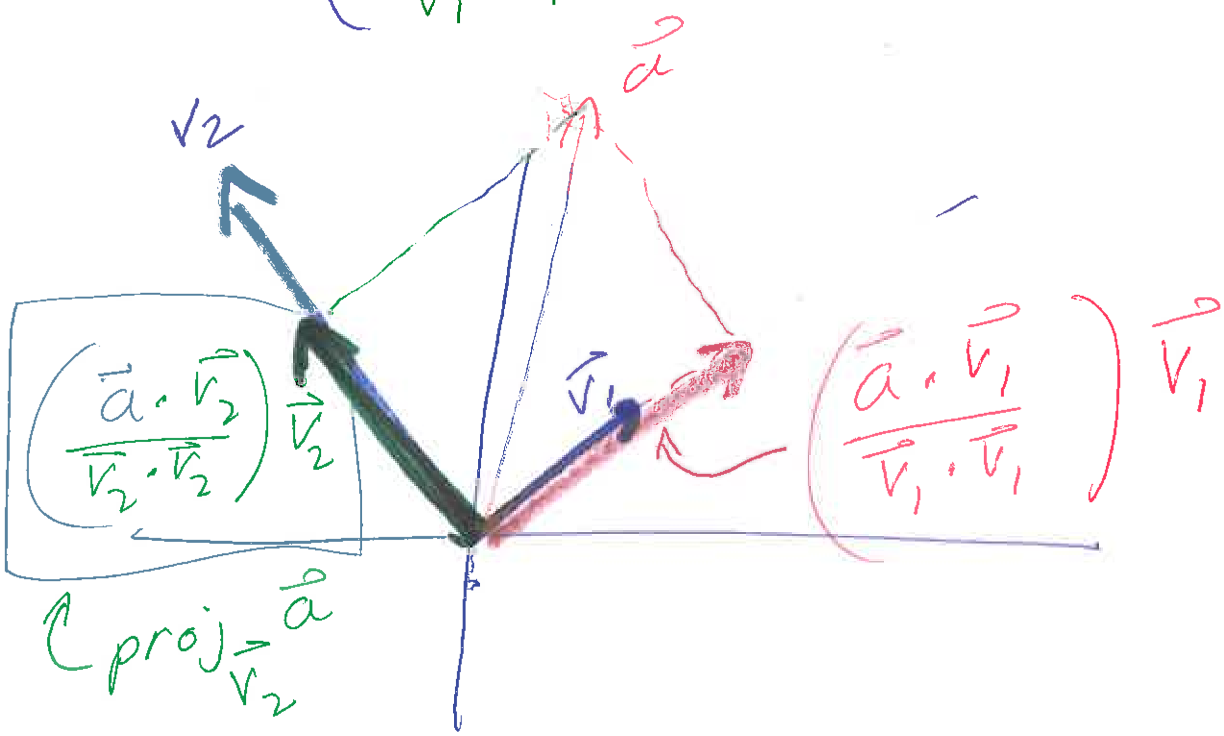
$$a \cdot \vec{v}_1 = c_1 \vec{v}_1 \cdot \vec{v}_1 + \cancel{c_2 \vec{v}_2 \cdot \vec{v}_1}$$

$$\frac{a \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = c_1$$



If $\vec{a}$ is in span $\{\vec{v}_1, \vec{v}_2\}$

$$\vec{a} = \left(\frac{\vec{a}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 + \left(\frac{\vec{a}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2$$

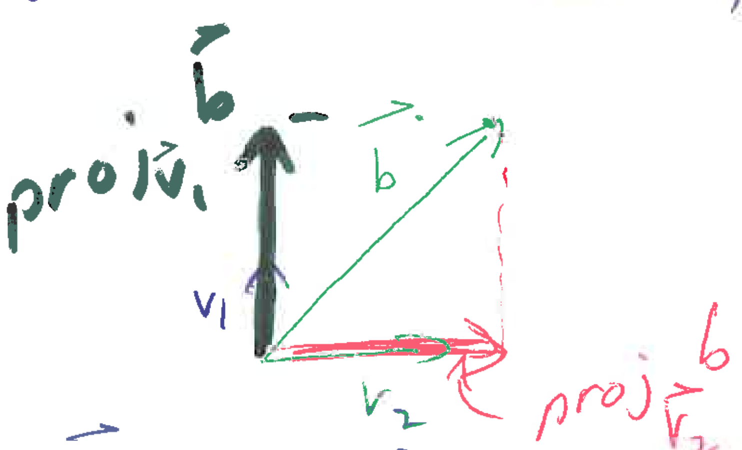


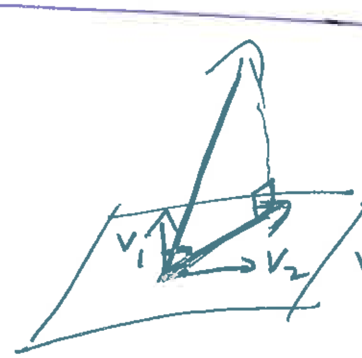
If $\vec{b}$ is not in span $\{v_1 \dots v_n\}$ where $\{v_1 \dots v_n\}$ is an orthogonal set

$$\operatorname{proj}_{\operatorname{span}\{v_1 \dots v_n\}} b = \sum_{i=1}^{m} c_i \vec{v_i}$$

$$c_i = \frac{\vec{b} \cdot \vec{v_i}}{\vec{v_i} \cdot \vec{v_i}}$$

$\vec{b}$ in span $\{v_1 \dots v_n\}$





v_1 & v_2 lie in plan

$$\vec{b} = \underbrace{\operatorname{proj}_{\vec{v}_1} \vec{b}}_{c_1 \vec{v}_1} + \underbrace{\operatorname{proj}_{\vec{v}_2} \vec{b}}_{c_2 \vec{v}_2}$$

$$\vec{b} = \underbrace{c_1 \vec{v}_1}_{\operatorname{proj}_{\vec{v}_1} \vec{b}} + \underbrace{c_2 \vec{v}_2}_{\operatorname{proj}_{\vec{v}_2} \vec{b}}$$

$\vec{v}_j \cdot \vec{a} = (c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_j \vec{v}_j + \dots + c_n \vec{v}_n) \cdot \vec{v}_j$

$\vec{v}_j \cdot \vec{a} = c_1 \cancelto{0}{(\vec{v}_1 \cdot \vec{v}_j)} + c_2 \cancelto{0}{(\vec{v}_2 \cdot \vec{v}_j)} + \dots + c_j (\vec{v}_j \cdot \vec{v}_j) + \dots + c_n (\vec{v}_n \cdot \vec{v}_j)$

$v_j \cdot a = c_j (v_j \cdot v_j) \Rightarrow c_j = \frac{\vec{v}_j \cdot \vec{a}}{\vec{v}_j \cdot \vec{v}_j}$

Thm: Let $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal basis for an inner product space V. Let $\mathbf{a}$ be an arbitrary vector in V. Then

$c_j = \frac{\vec{a} \cdot \vec{v}_j}{\vec{v}_j \cdot \vec{v}_j}$

$$\mathbf{a} = c_1 \mathbf{v_1} + ... + c_n \mathbf{v_n}$$

$$\text{where } \boxed{c_j = \frac{<\mathbf{a}, \mathbf{v_j}>}{||\mathbf{v_j}||^2}} \text{ for } j = 1, 2, ..., n.$$

$[\vec{v}_1 \cdots \vec{v}_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$

$(k \times n)(n \times 1) \quad (k \times 1)$

$A \vec{x} = \vec{b}$

$\vec{V} \vec{c} = \vec{a}$

Note if $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is an orthonormal basis, then

$$||\mathbf{v_j}|| = 1 \text{ and } c_j =< \mathbf{a}, \mathbf{v_j} >$$

Thm: Let $\mathbf{a}$, $\mathbf{v}$ be nonzero vectors in R^k.

The vector component of $\mathbf{a}$ along $\mathbf{v}$

$$= \text{orthogonal projection of } \mathbf{a} \text{ on } \mathbf{v}$$
$$= proj_{\mathbf{v}} \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{v}}{||\mathbf{v}||^2} \mathbf{v}$$

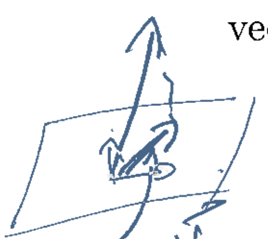
$\vec{a} = \boxed{c_1 \vec{V}} + c_2 \vec{w} + c_3 \vec{u}$

$c_1 = \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$

The vector component of $\mathbf{a}$ orthogonal to $\mathbf{v}$

$$= \mathbf{a} - proj_{\mathbf{v}} \mathbf{a} = \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{v}}{||\mathbf{v}||^2} \mathbf{v}$$

Thm: Let $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be an orthogonal basis for subspace W of an inner product space V. Let $\mathbf{a}$ be an arbitrary vector in V. Then



$c_1 \vec{v}_1 + c_2 \vec{v}_2$

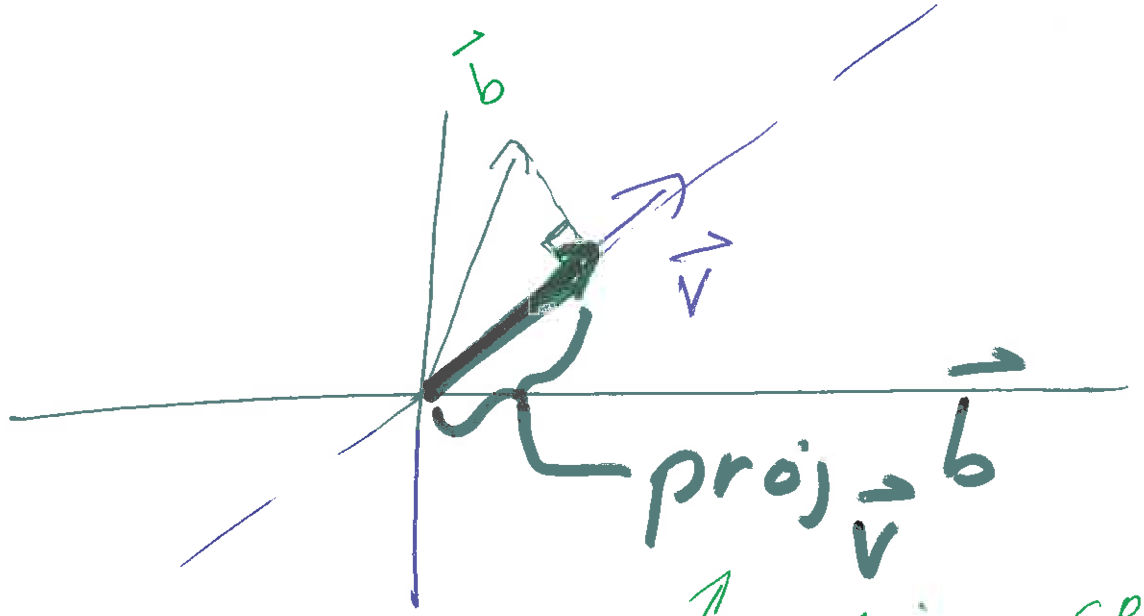
$$proj_W \mathbf{a} = c_1 \mathbf{v_1} + ... + c_n \mathbf{v_n}$$

$$\text{where } c_j = \frac{<\mathbf{a}, \mathbf{v_j}>}{||\mathbf{v_j}||^2} \text{ for } j = 1, 2, ..., n.$$

Note if $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ is an orthonormal basis, then

$$||\mathbf{v_j}|| = 1 \text{ and } c_j =< \mathbf{a}, \mathbf{v_j} >$$

The vector component of $\mathbf{a}$ orthogonal to $\mathbf{W} = \mathbf{a} - proj_{\mathbf{W}} \mathbf{a}$



$$\mathrm{proj}_{\vec{v}}\,\vec{b} = \left(\frac{\vec{b}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

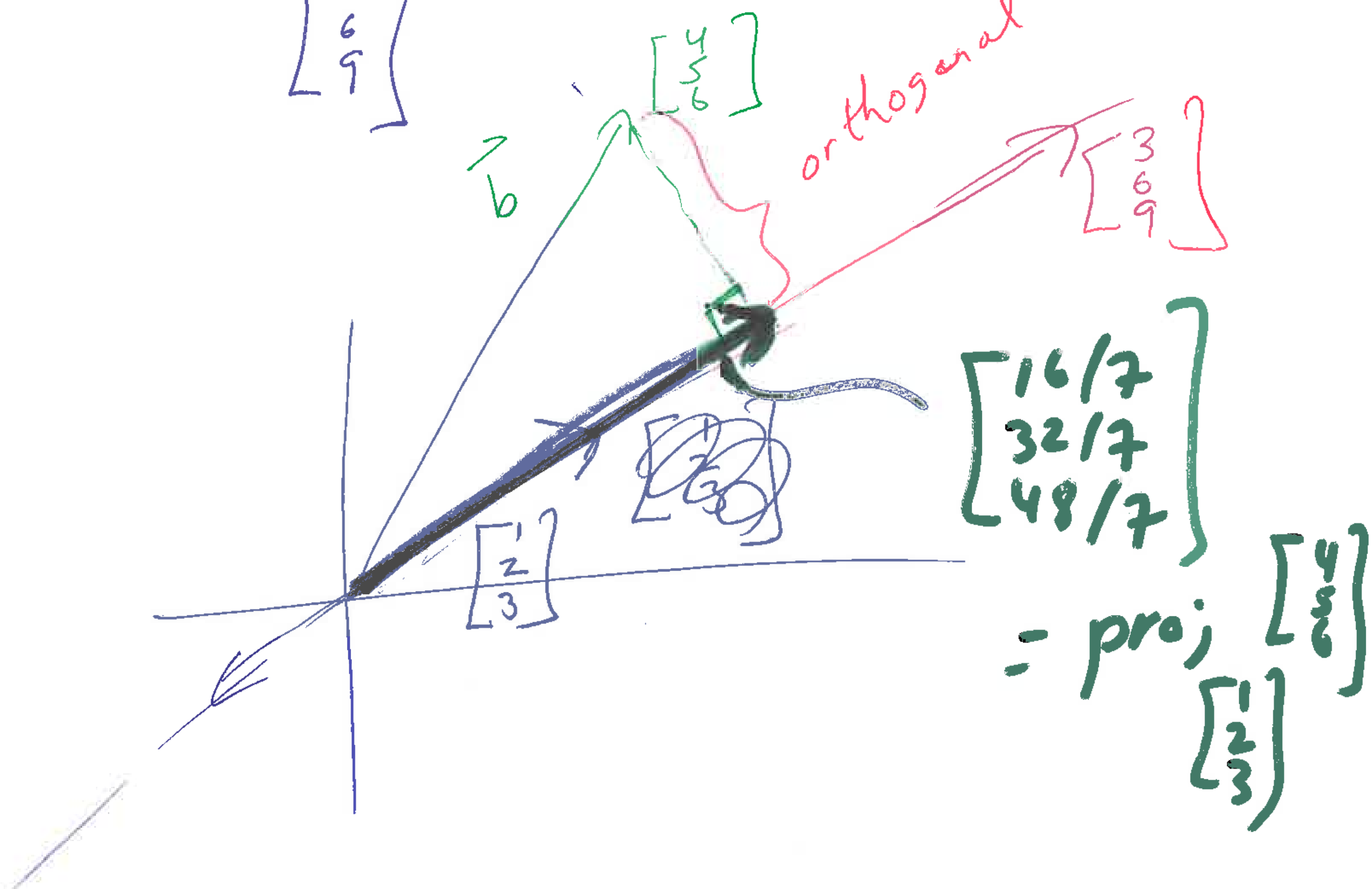
direction

Ex: $\mathrm{proj}_{\begin{bmatrix}1\\2\\3\end{bmatrix}}\begin{bmatrix}4\\5\\6\end{bmatrix} = \dfrac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}}{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}}\begin{bmatrix}1\\2\\3\end{bmatrix}$

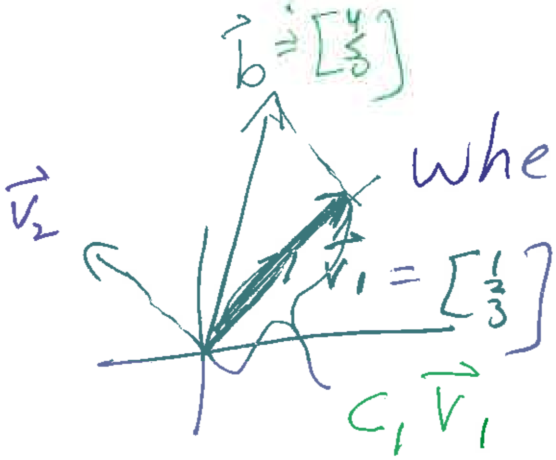
$$= \frac{4+10+18}{1+4+9}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{32}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\begin{bmatrix}16/7\\32/7\\48/7\end{bmatrix}$$

$$\mathrm{proj}_{\begin{bmatrix}3\\6\\9\end{bmatrix}}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}16/7\\32/7\\48/7\end{bmatrix}$$



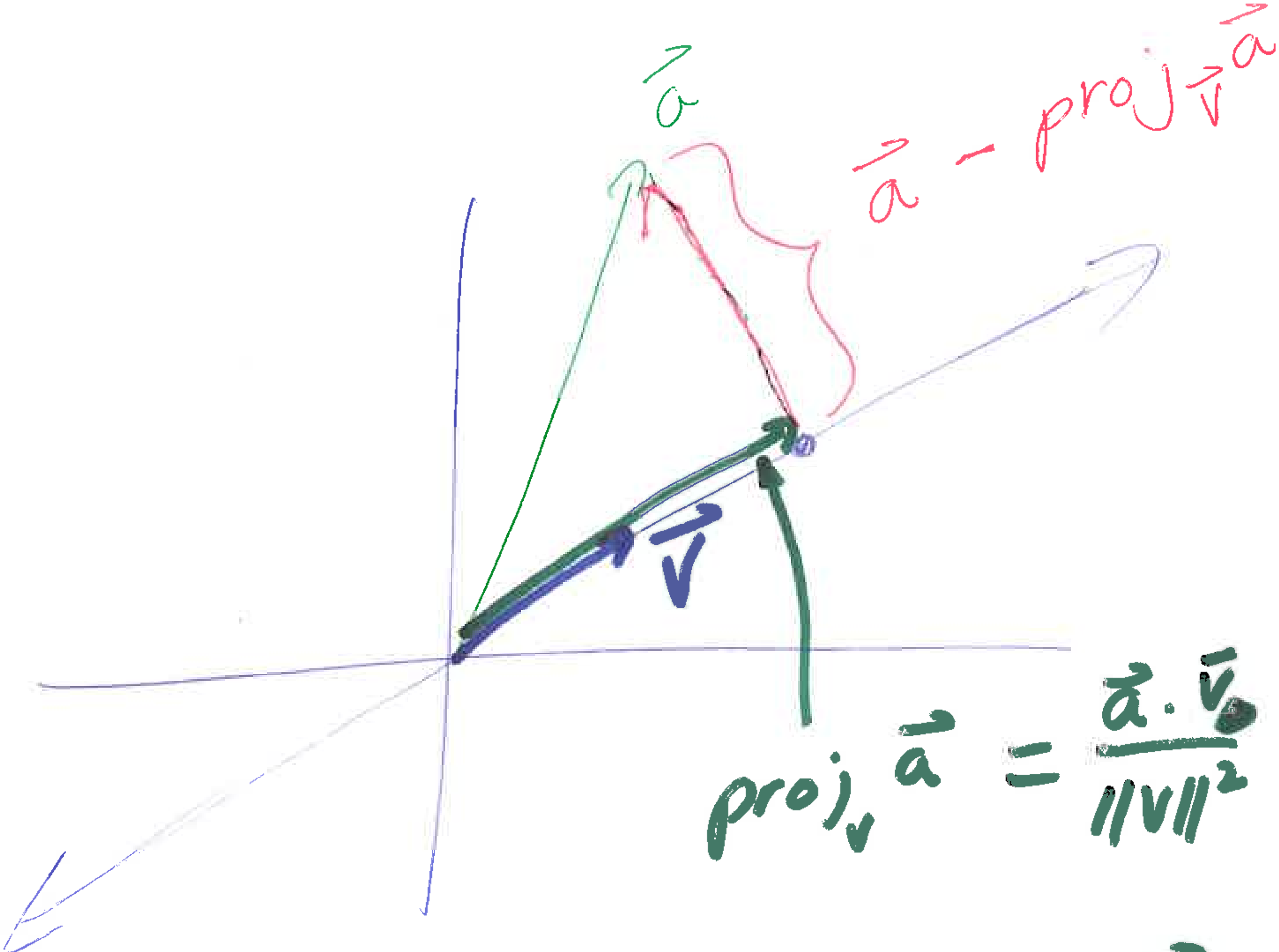
$$\begin{bmatrix}4\\5\\6\end{bmatrix} = c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\vec{v_2} + c_3\vec{v_3}$$



where

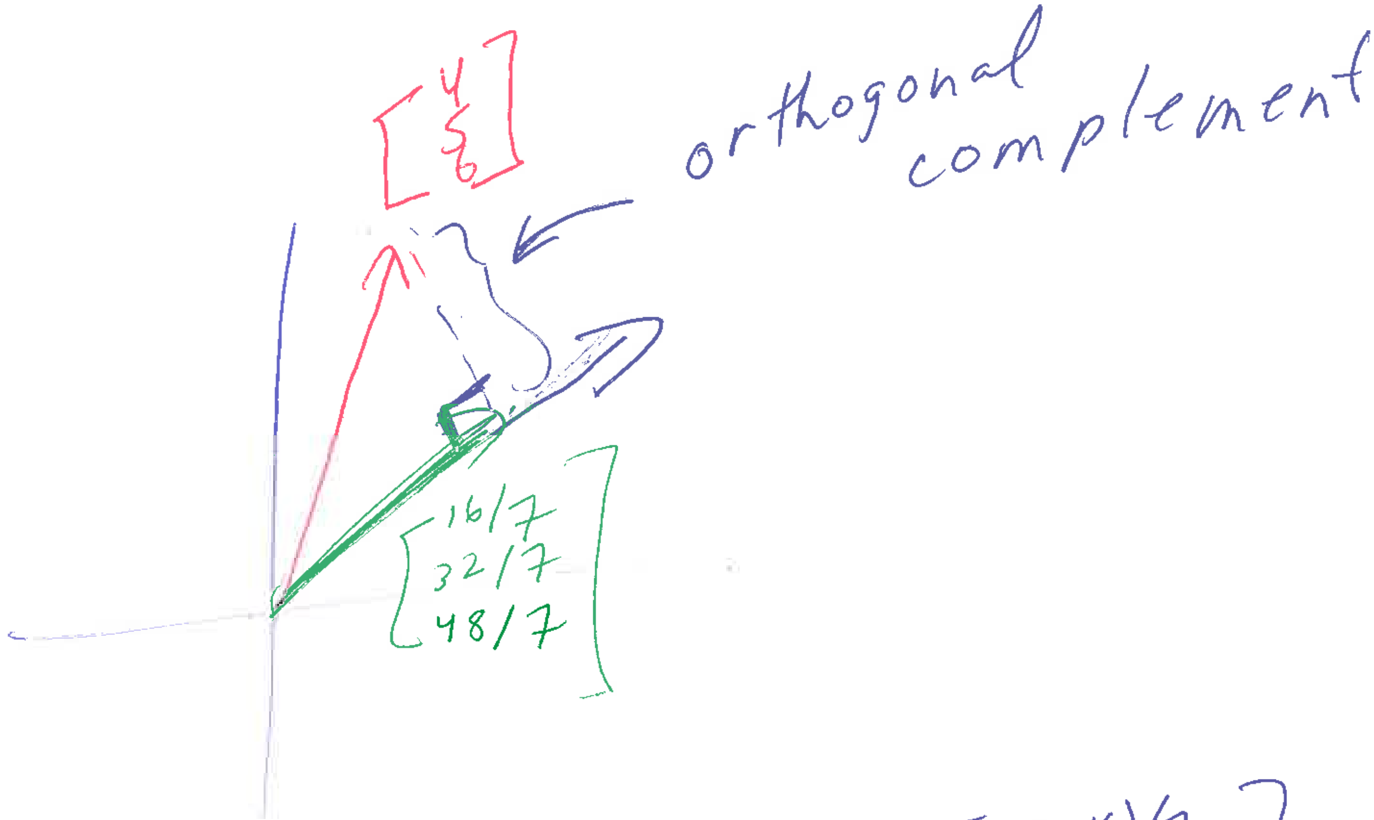
$$c_1 = \frac{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}}{\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}}$$

$\mathrm{proj}_{\vec{v_1}} \vec{b}$



$$\mathrm{proj}_{v}\vec{a} = \frac{\vec{a}\cdot\vec{v}}{\|v\|^2}\vec{v}$$

$$= \frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}$$



$$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 16/7 \\ 32/7 \\ 48/7 \end{bmatrix} = \begin{bmatrix} (28-16)/7 \\ (35-32)/7 \\ (42-48)/7 \end{bmatrix}$$

vector component of $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ orthogonal $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12/7 \\ 3/7 \\ -6/7 \end{bmatrix}$

Check:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 12/7 \\ 3/7 \\ -6/7 \end{bmatrix} = \frac{12}{7} + \frac{6}{7} - \frac{18}{7}$$

$$= 0 \checkmark$$

$\text{span}\{\vec v_1, \vec a_2\} = \text{span}\{\vec v_1, \vec v_2\}$

$\text{proj}_{\vec v_1} \vec a_2 + \vec v_2 = \vec a_2$

Thm (Gram-Schmidt process for constructing an orthogonal basis):

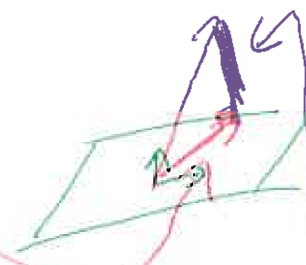
Let $\mathcal{T} = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ be a basis for an inner product space V. Let $\mathcal{T}' = \{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ be defined as follows:

$$\mathbf{v_1} = \mathbf{a_1}$$

$$\mathbf{v_2} = \mathbf{a_2} - \frac{<\mathbf{a_2}, \mathbf{v_1}>}{<\mathbf{v_1}, \mathbf{v_1}>}\mathbf{v_1}$$

$$\mathbf{v_3} = \mathbf{a_3} - \frac{<\mathbf{a_3}, \mathbf{v_1}>}{<\mathbf{v_1}, \mathbf{v_1}>}\mathbf{v_1} - \frac{<\mathbf{a_3}, \mathbf{v_2}>}{<\mathbf{v_2}, \mathbf{v_2}>}\mathbf{v_2}$$

$\text{span}\{V_1, V_2\}$



$$\vdots$$

$a_3 - \text{proj}_{\text{span}\{v_1, v_2\}} a_j$

$$\mathbf{v_n} = \mathbf{a_n} - \frac{<\mathbf{a_n}, \mathbf{v_1}>}{<\mathbf{v_1}, \mathbf{v_1}>}\mathbf{v_1} - \frac{<\mathbf{a_n}, \mathbf{v_2}>}{<\mathbf{v_2}, \mathbf{v_2}>}\mathbf{v_2} - ... - \frac{<\mathbf{a_n}, \mathbf{v_{n-1}}>}{<\mathbf{v_{n-1}}, \mathbf{v_{n-1}}>}\mathbf{v_{n-1}}$$

Then the set $\mathcal{T}'$ is an orthogonal basis for V.

$v_n = \vec a_n - \text{proj}_{\text{span}\{v_1, \dots v_{n-1}\}} \vec a_n$

An orthonormal basis for V is given by

$$\mathcal{T}'' = \left\{ \frac{\mathbf{v_1}}{||\mathbf{v_1}||}, \frac{\mathbf{v_2}}{||\mathbf{v_2}||}, ..., \frac{\mathbf{v_n}}{||\mathbf{v_n}||} \right\}.$$

$\vec v_n$ = orthog complement of $\vec a_n$ onto $\text{span}\{V_1 \dots V_{n-1}\}$