To determine if $\vec{v}$ is an e. vector, calculate $A\vec{v}$

$\boxed{\text{Check if } A\vec{v} \text{ is a multiple of } \vec{v}}$

$\boxed{A\vec{v} = \lambda \vec{v}}$

5.1: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the matrix A if there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$.

$A\vec{v} = -1\vec{v}$

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad A\vec{w} = 5\vec{w}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is NOT an eigenvector of A.

not a multiple of $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

1) $\vec{0}$ is never an ~~e. value~~ by definition e. vector

$$A\vec{0} = \vec{0} = \lambda\vec{0}$$

for all λ

We want λ to be unique to its eigen vector

I.e. An e. vector corresponds to exactly 1 e. value

Determine if $(3, -2)$ is an e. vector of

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

Calculate Av

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is an e. vector w/ e. value 0

Note 0 can be an e. value

But $\vec{0}$ is never an e. vector

Is $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ an e. vector of $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 26 \end{bmatrix} \neq \lambda \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ is Not an e. vector of $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

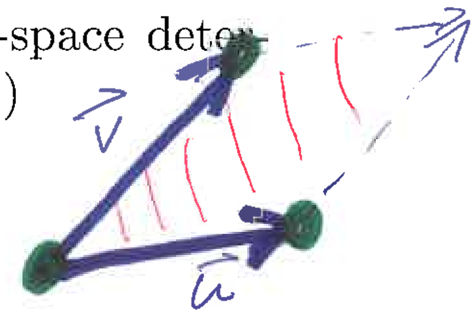
Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ an e. vector of $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

NO!

Area and Volume

a.) The area of the parallelogram in 2-space determined by the vectors (u_1, u_2) and (v_1, v_2)

$$\text{abs} = \left| det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$



b.) The volume of the parallelepiped in 3-space determined by the vectors (u_1, u_2, u_3), (v_1, v_2, v_3), and (w_1, w_2, w_3)

$$\text{abs} = \left| det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

Example: Find the area of the parallelogram determined by the vectors (1, 2) and (3, 4).

$$\text{abs}\left(\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \right) = \text{abs}(4 - 6) = \text{abs}(-2) = +2$$

$$\text{abs}\left(\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \right) = \text{abs}(4 - 6) = \text{abs}(-2) = +2$$

Example: Find the ~~area~~ volume of the parallelepiped determined by vectors (1, 4, 5), (2, 10, 0), & (3, 0, 6)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5^{-2} & 0^{-4} & 6^{-6} \end{vmatrix} \xrightarrow[\to 1R_3]{1R_3 - 2R_1} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 3 & -4 & 0 \end{vmatrix} = 3 \begin{vmatrix} 4 & 10 \\ 3 & -4 \end{vmatrix}$$

$- 0 + 0$

$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 2 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix}$$

$$|| \quad (1R_2 - R_1 \rightarrow 1R_2), (1R_4 - 2R_1 \rightarrow 1R_4)$$

$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & -4 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & -1 & -4 & -2 \end{bmatrix}$$

$$|| \quad (R_2 \leftrightarrow R_4)$$

minus

$$-det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$|| \quad (R_3 + 3R_2 \rightarrow R_3)$$

$$-det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & -14 & -5 \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$\frac{R3}{-14}$$

$$|| \quad (\tfrac{-1}{14} R_3 \rightarrow R_3)$$

$$-(-14) det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & -4 & -1 \end{bmatrix}$$

$$|| \quad (1R_4 + 4R_3 \rightarrow 1R_4)$$

$$14 det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 1 & \frac{5}{14} \\ 0 & 0 & 0 & \frac{3}{7} \end{bmatrix} = 14(2)(-1)(1)(\tfrac{3}{7}) = -12$$

$1R_4 + 4R_3 \rightarrow 1R_4$

$1R_i - hR_j \rightarrow 1R_i$

Don't change determinant

$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 2 & 1 & -1 & 1 \\ 0 & 3 & -2 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix}$$

$$|| \quad (R_2 - R_1 \to R_2), (R_4 - 2R_1 \to R_4)$$

$$det \begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & -4 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & -1 & -4 & -2 \end{bmatrix} = (-1)^{1+1} 2det \begin{bmatrix} 0 & -4 & -1 \\ 3 & -2 & 1 \\ -1 & -4 & -2 \end{bmatrix}.$$

$R_2 + 3R_3 \to R_2$

$$|| \quad (\cancel{R_2 + 3R_3 \to R_3})$$

2

$$2det \begin{bmatrix} 0 & -4 & -1 \\ 0 & -14 & -5 \\ -1 & -4 & -2 \end{bmatrix}$$

$$||$$

$$-12 = 2[(-1)\{20 - 14\}] = 2[(-1)^{1+3}(-1)\{(-4)(-5) - (-14)(-1)\}]$$

A H

Suppose $det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 5$ and $det \begin{bmatrix} e & f \\ g & h \end{bmatrix} = 2$

$R_1 \leftrightarrow R_2$

3 (col 1)
4 A'

Then $det(4 \begin{bmatrix} 3a & c \\ 3b & d \end{bmatrix} \begin{bmatrix} g & h \\ e & f \end{bmatrix}) = det(\begin{bmatrix} 12a & 4c \\ 12b & 4d \end{bmatrix} \begin{bmatrix} g & h \\ e & f \end{bmatrix})$

$$= det \begin{bmatrix} 12a & 4c \\ 12b & 4d \end{bmatrix} det \begin{bmatrix} g & h \\ e & f \end{bmatrix} = 4^2 det \begin{bmatrix} 3a & c \\ 3b & d \end{bmatrix} det \begin{bmatrix} g & h \\ e & f \end{bmatrix}$$

T

$$= 4^2 det \begin{bmatrix} 3a & 3b \\ c & d \end{bmatrix} det \begin{bmatrix} g & h \\ e & f \end{bmatrix} = 3 \times 4^2 det \begin{bmatrix} a & b \\ c & d \end{bmatrix} det \begin{bmatrix} g & h \\ e & f \end{bmatrix}$$

$$= -3 \times 4^2 det \begin{bmatrix} a & b \\ c & d \end{bmatrix} det \begin{bmatrix} e & f \\ g & h \end{bmatrix} = -3 \times 4^2 \times 5 \times 2 = -480$$

A H

$$-4^2 \cdot 3(5)(2) = -480$$

Thm 8′: If A is a SQUARE $n \times n$ matrix, then the following are equivalent.

a.) A is invertible. $A \sim I_n$

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries [I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row).

d.) The column vectors of A are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any b.

g.) $Ax = b$ has a unique sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n.
[every vector in R^n can be written as a linear combination of the columns of A].

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

m.) A^T is invertible.

n.) A is expressible as a product of elementary matrices.

o.) The column vectors of A form a basis for R^n.
[every vector in R^n can be written uniquely as a linear combination of the columns of A].

p.) Col $A = R^n$.

q.) dim Col $A = n$.

r.) rank of $A = n$.

s.) Nul $A = \{\mathbf{0}\}$,

t.) dim Nul $A = 0$.

u.) A has nullity 0.

Det $A \neq 0$

Rank(A) + nullity(A) = Number of columns of A.

Ex. 2) Suppose A is a $9X4$ matrix.

If Rank(A) = 4, then nullity(A) =

$A\mathbf{x} = \mathbf{0}$ has ________________ solutions.

$A\mathbf{x} = \mathbf{b}$ has ________________ solutions.

If Rank(A) = 3, then nullity(A) =

$A\mathbf{x} = \mathbf{0}$ has ________________ solutions.

$A\mathbf{x} = \mathbf{b}$ has ________________ solutions.

$A\vec{x} = 0$

$(h \times n)(n \times 1) = h \times 1$

$\text{Nul } A \subset \mathbb{R}^n$

Nullspace of A = solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace:

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{0}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is also a solution: $\Rightarrow A\vec{v}_1 = 0$ $\quad A\vec{v}_2 = 0$

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2$$
$$= c_1(\vec{0}) + c_2(\vec{0}) = \vec{0}$$

The solution set of $A\mathbf{x} = \mathbf{b}$ is NOT a subspace unless $\mathbf{b} = \mathbf{0}$:

pf: $A\vec{0} = \vec{0} \neq \vec{b}$ thus $\vec{x} = 0$ is not in solution set to $A\vec{x} = \vec{b}$. Thus not a subspace

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is a solution to $(c_1 + c_2)\vec{b}$

Not closed under linear combinations

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1(A\vec{v}_1) + c_2(A\vec{v}_2)$$
$$= c_1\vec{b} + c_2\vec{b} = (c_1 + c_2)\vec{b} \neq \vec{b}$$

unless $\vec{b} = 0$ or $c_1 + c_2 = 1$

Ch 5: The eigenspace corresponding to an eigenvalue λ is a subspace.

Determine the nullspace of B where B $\sim$ $\begin{array}{ccccc} \left[\begin{matrix} 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}\right] \\ x_1 & & & x_4 & \end{array}$

Solve $Bx=0$

Need REF

Solve: $B\mathbf{x} = \mathbf{0}$ where $B \sim \begin{array}{cccccc} \left[\begin{matrix} 0 & 1 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}\right] \\ x_1 & & & x_4 & & \end{array}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -8x_4 \\ -6x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -8x_4 \\ -6x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -8 \\ -6 \\ 1 \\ 0 \end{bmatrix} x_4$$

Solve: $B\mathbf{x} = \mathbf{0}$ where $B \sim \begin{array}{cccccc} \left[\begin{matrix} 0 & 1 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}\right] \\ x_1 & & & x_4 & & \end{array}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -8 \\ -6 \\ 1 \\ 0 \end{bmatrix} x_4$$

Solve: $B\mathbf{x} = \mathbf{0}$ where $B \sim \begin{array}{cccccc} \left[\begin{matrix} 0 & 1 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}\right] \\ x_1 & & & x_4 & & \end{array}$

$$\text{Nul } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -8 \\ -6 \\ 1 \\ 0 \end{bmatrix} x_4 \;\middle|\; x_1,\ x_4 \text{ in } R \right\}$$

Solve: $B\mathbf{x} = \mathbf{0}$ where $B \sim \begin{array}{cccccc} \left[\begin{matrix} 0 & 1 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}\right] \\ x_1 & & & x_4 & & \end{array}$

$$\text{Nul } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -8 \\ -6 \\ 1 \\ 0 \end{bmatrix} x_4 \;\middle|\; x_1,\ x_4 \text{ in } R \right\}$$

Solve: $B\mathbf{x} = \mathbf{0}$ where $B \sim \begin{array}{cccccc} \left[\begin{matrix} 0 & 1 & 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}\right] \\ x_1 & & & x_4 & & \end{array}$

$$\text{Nul } B = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ -6 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Basis for Nul B

REF

Solve: $E\mathbf{x} = \mathbf{0}$ where $E \sim \begin{bmatrix} 0 & 1 & 0 & -5 & 0 & 0 & 5 \\ 0 & 0 & 1 & 7 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_1 \\ 5x_4 - 5x_7 \\ -7x_4 + 3x_7 \\ x_4 \\ 0 \\ x_7 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 5 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ -5 \\ 3 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} x_7$$

Nul E = $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 3 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

basis

Determine the column space of E where $E \sim \begin{bmatrix} 0 & 1 & 0 & -5 & 0 & 0 & 5 \\ 0 & 0 & 1 & 7 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

IF POSSIBLE

Note: We don't know the original matrix E. We only know REF of E.

NOT POSSIBLE

NOT ENOUGH INFO

Row ops affect the column space

Determine the column space of B where $B \sim \begin{bmatrix} 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

IF POSSIBLE

Note: We don't know the original matrix B. We only know REF of B.

But pivot in each row

Span $= \mathbb{R}^3$

Determine the column space of $A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Column space of A = col A =

col A = $\text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix}, \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix}, \cancel{\begin{bmatrix} -42 \\ -32 \\ 87 \end{bmatrix}} \right\}$

$= \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix} + c_3 \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} + \cancel{c_4 \begin{bmatrix} -42 \\ -32 \\ 87 \end{bmatrix}} \;\middle|\; c_i \text{ in } \mathbb{R} \right\}$

NOT SIMPLIFIED

Determine the column space of $A = \begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix}$

Put A into echelon form:

$$\begin{bmatrix} 1 & -10 & -24 & -42 \\ 1 & -8 & -18 & -32 \\ -2 & 20 & 51 & 87 \end{bmatrix} \xrightarrow[R_3 + 2R_1 \to R_3]{R_2 - R_1 \to R_2} \begin{bmatrix} 1 & -10 & -24 & -42 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

REF

A basis for col A consists of the 3 pivot columns from the **original** matrix A.

Thus basis for col A = $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} -10 \\ -8 \\ 20 \end{bmatrix} \begin{bmatrix} -24 \\ -18 \\ 51 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

span $= \mathbb{R}^3$

Defn: Let W be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for W if

i.) $\mathcal{T}$ is linearly independent and ← simplified

ii.) $\mathcal{T}$ spans W. ← describes W

I.e.,

$\mathcal{T}$ is the smallest collections of vectors that span W.

Basis thm: Let W be a p-dimensional subspace of R^n.

i.) If $W = \text{span}\{w_1, ..., w_p\}$, then $\{w_1, ..., w_p\}$ is a basis for W.

span $\Rightarrow$ l.i. since $p = p$

ii.) If $v_1, ..., v_p$ are linearly independent vectors in W, then $\{v_1, ..., v_p\}$ is a basis for W. $\Rightarrow$ p dimensional span

li $\Rightarrow$ span

Thm: All basis for a finite-dimensional vector space have the same number of elements.

Defn:

$dim(V)$ = the **dimension** of a finite-dim vector sp V
= the number of vectors in any basis for V.

If $dim(V) = n$, then V is said to be n-dimensional.

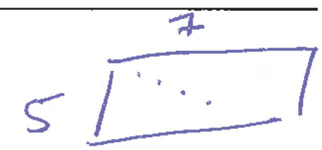
rank A = Rank of a matrix A = dimension of Col A
= number of pivot columns of A.

nullity of A = dimension of Nul A
= number of free variables.

Rank(A) + nullity(A) = Number of columns of A.

That is,

The number of pivots of A	+	The number of free variables of A	=	The number of columns of A



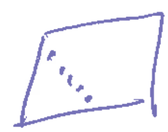
Ex. 1) Suppose A is a $5X7$ matrix.

If Rank(A) = 4, then nullity(A) = $7-4=3$ f. v.

$A\mathbf{x} = \mathbf{0}$ has ∞ solutions.

$A\mathbf{x} = \mathbf{b}$ has ∞ or none solutions.

If Rank(A) = 5, then nullity(A) = $7-5=2$ fv



$A\mathbf{x} = \mathbf{0}$ has ∞ solutions.

$A\mathbf{x} = \mathbf{b}$ has ∞ solutions. pivot in each row

If Rank(A) = 5, the column space of A = $\mathbb{R}^5$