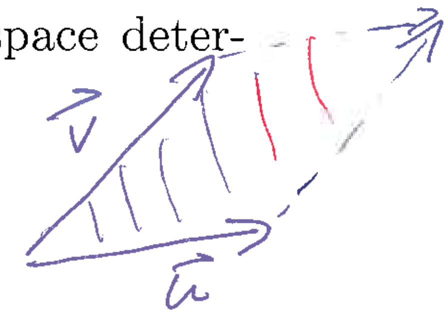
Area and Volume

a.) The area of the parallelogram in 2-space determined by the vectors (u_1, u_2) and (v_1, v_2)

$$= \left| det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$



b.) The volume of the parallelepiped in 3-space determined by the vectors (u_1, u_2, u_3), (v_1, v_2, v_3), and (w_1, w_2, w_3)

$$= \left| det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

Example: Find the area of the parallelogram determined by the vectors (1, 2) and (3, 4).

$$abs\left(\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\right) = abs(4 - 6) = abs(-2)$$
$$= +2$$

$$abs\left(\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}\right) = abs(4 - 6) = abs(-2)$$
$$= +2$$

Example: Find the ~~area~~ volume of the parallelepiped determined by vectors (1, 4, 5), (2, 10, 0), & (3, 0, 6)

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 5^{-2} & 0^{-4} & 6^{-6} \end{vmatrix} \xrightarrow{1R_3 - 2R_1 \to R_3} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 10 & 0 \\ 3 & -4 & 0 \end{vmatrix} = 3 \begin{vmatrix} 4 & 10 \\ 3 & -4 \end{vmatrix}$$

$$- 0 + 0$$

$= 3(-16-30) = 3(-46)$

$=$ [illegible]

$= -138$

Volume $= \text{abs}(-138)$

$= +138$

Recall how row operations affect the determinant:

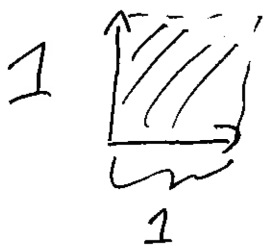
If $A \xrightarrow[R_i \to cR_i]{} B$, then $detB = c(detA)$.

If $A \xrightarrow[R_i \leftrightarrow R_j]{} B$, then $detB = -(detA)$.

If $A \xrightarrow[R_i + cR_j \to R_i]{} B$, then $detB = detA$.

Note how row operations affect area:

Area of square determined by vectors (1, 0) & (0, 1):

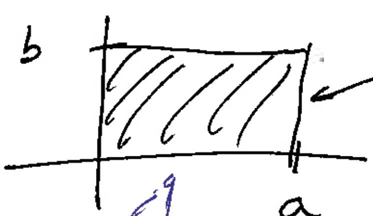


$$= 1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$aR_1 \to R_1$

$bR_2 \to R_2$

Area of rectangle determined by vectors (a, 0) & (0, b): [redacted]



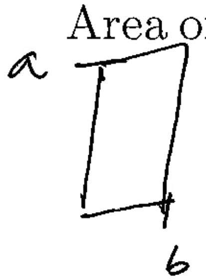
$$|ab| = \text{abs} \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$R_2 + 3R_1 \to R_2$

Area of rectangle determined by vectors (a, 3a) & (0, b): [redacted]

$$|ab| = \text{abs} \begin{vmatrix} a & 0 \\ 3a & b \end{vmatrix}$$

Area of rectangle determined by vectors (0, a) & (b, 0): [redacted]



$$= \text{abs}(ab) = \text{abs} \begin{vmatrix} 0 & b \\ a & 0 \end{vmatrix}$$

Ch 2 partial review:

Recall W is a **subspace** of R^n (**vector space**) if W is closed under scalar multiplication and vector addition.

I.e., W is a subspace of R^n if

$$\mathbf{v_1}, \mathbf{v_2} \text{ in } W \text{ implies } c_1\mathbf{v_1} + c_2\mathbf{v_2} \text{ in } W.$$

← closed under linear combination

Note if W is a finite dimensional subspace, ~~then~~ iff for some vectors $\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}$ in W:

$$W = span\{\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}\}$$

$$= \{c_1\mathbf{w_1} + c_2\mathbf{w_2} + ... + c_k\mathbf{w_k} \mid c_i \in R\}$$

= the set of all linear combinations of the vectors $\mathbf{w_1}, \mathbf{w_2}, ..., \mathbf{w_k}$.

Examples: A $k \times n$ ← k rows $\quad$ col $A \subset R^K$

The column space of $A = [\mathbf{a_1}\ \mathbf{a_2}\ ...\ \mathbf{a_n}]$

$$= \{c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} \mid c_i \in R\}$$

$$= \{\mathbf{b} \mid A\mathbf{x} = \mathbf{b} \text{ has at least one solution } \}$$

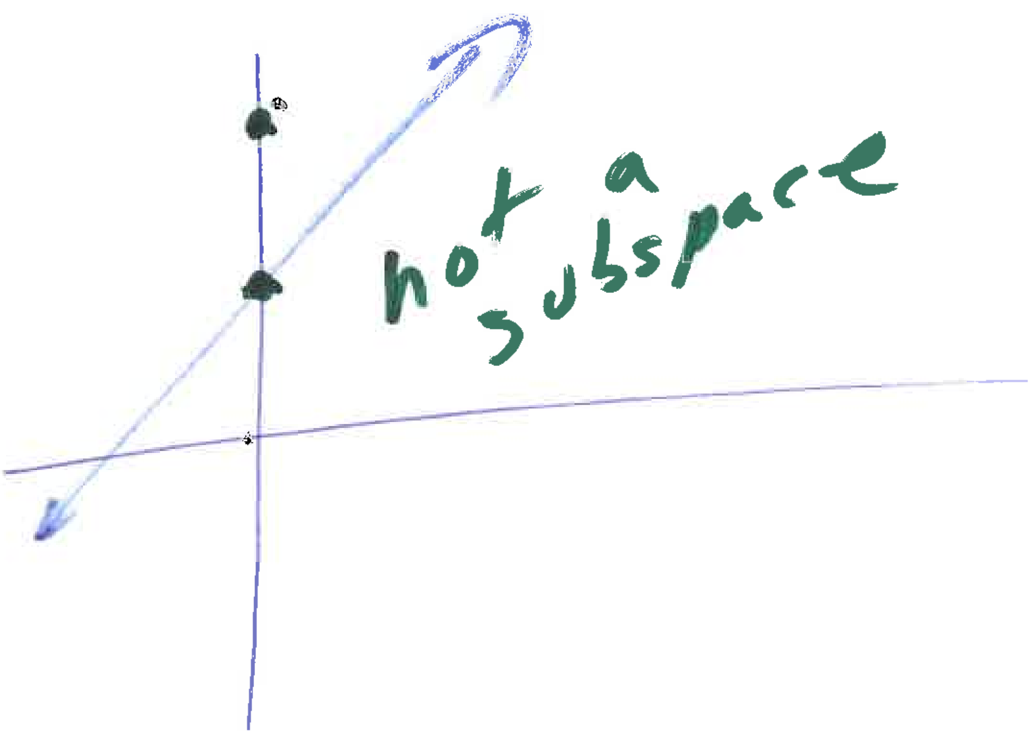
is a subspace.

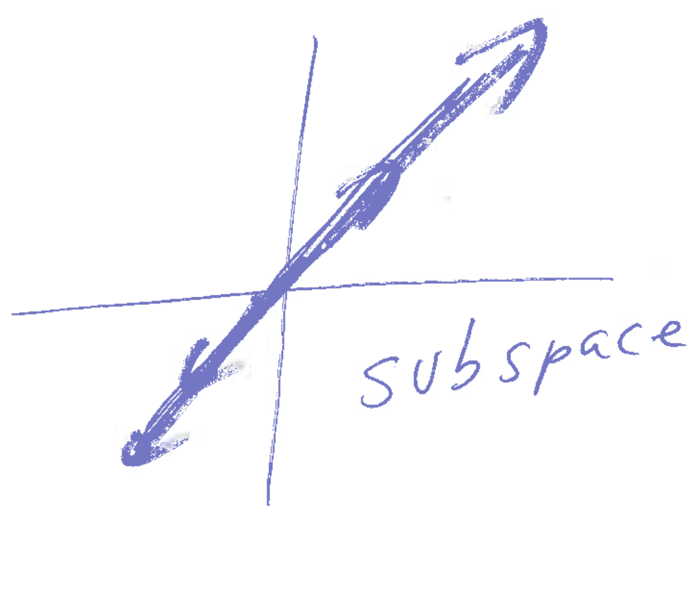
$$\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix}$$

$(K \times n)$ $\quad (n \times 1)$ $\quad (K * 1)$

negative #'s are not closed under mult

$(-2)(-3) = +6$





$$\boxed{\mathrm{col}(A) = \mathrm{span}\{a_1 \dots a_n\}}$$

$$= \{ c_1 \vec{a}_1 + \dots + c_n \vec{a}_n \mid c_i \in \mathbb{R} \}$$

$$= \{ \mathbf{b} \mid Ax = \mathbf{b} \text{ st there exists } c \text{ s.t. } Ac = b \}$$

$$= \{ \mathbf{b} \mid Ac = \mathbf{b} \text{ for some solution } c \}$$

b is in col A

$\iff$ $Ax = b$ has a sol'n

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\text{col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cancel{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= \mathbb{R}^2$$

Is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in col A?

Does $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

have a soln

$A\vec{x} = 0$

$(h \times n)(n \times 1) = h \times 1$

$\text{Nul } A \subset \mathbb{R}^n$

Nullspace of A = solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace:

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{0}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is also a solution: $\Rightarrow A\vec{v}_1 = 0$ $\quad A\vec{v}_2 = 0$

$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2$

$= c_1(\vec{0}) + c_2(\vec{0})$

$= \vec{0}$

The solution set of $A\mathbf{x} = \mathbf{b}$ is NOT a subspace unless $\mathbf{b} = \mathbf{0}$:

pf: $A\vec{0} = \vec{0} \neq \vec{b}$ thus $\vec{x} = 0$ is not in solution set to $A\vec{x} = \vec{b}$. Thus not a subspace

If $\mathbf{v_1}, \mathbf{v_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$, then $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ is a solution to $(c_1 + c_2)\vec{b}$

$A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1(A\vec{v}_1) + c_2(A\vec{v}_2)$

$= c_1\vec{b} + c_2\vec{b} = (c_1 + c_2)\vec{b} \neq \vec{b}$

unless $\vec{b} = 0$ or $c_1 + c_2 = 1$

Ch 5: The eigenspace corresponding to an eigenvalue λ is a subspace.

To determine if $\vec{v}$ is an e. vector, calculate $A\vec{v}$

$\boxed{\text{Check if } A\vec{v} \text{ is a multiple of } \vec{v}}$

$$\boxed{A\vec{v} = \lambda \vec{v}}$$

5.1: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the matrix A if there exists a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.

The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

Example: Let $A = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$.

$A\vec{v} = -1\vec{v}$

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\vec{w} = 5\vec{w}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is NOT an eigenvector of A.

not a multiple of $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$

not an eigenvector

eigenvectors

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A\left(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $A\mathbf{x} - \lambda I\mathbf{x} = (A - \lambda I)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(A - \lambda I)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Claim $S = \{ \vec{x} \mid A\vec{x} = \lambda \vec{x} \}$

is a subspace for fixed λ

Pf: Suppose $\vec{v}$ and $\vec{w}$ are in S

$A\vec{v} = \lambda \vec{v}$ $\quad\quad$ $A\vec{w} = \lambda \vec{w}$

$$A(c_1\vec{v} + c_2\vec{w}) = c_1(A\vec{v}) + c_2(A\vec{w})$$
$$= c_1(\lambda\vec{v}) + c_2(\lambda\vec{w})$$
$$= \lambda(c_1\vec{v} + c_2\vec{w})$$

The eigen space for A corresponding to e. value λ

is a subspace

Since we have just show it is closed under linear combinatn

The e. space for A corresponding to e. value λ

$$= \{ \vec{X} \mid A\vec{X} = \lambda \vec{X} \}$$

λ fixed

is a subspace

Notation warning

$\vec{O}$ is in the e. space

But $\vec{O}$ is never an e. vector of A by defn

$A\vec{O} = \lambda \vec{O}$ for all real #'s λ

We want λ to be unique

The e. space of A corresponding to e. value λ

consists of all e. vectors of A w/ e. value λ

PLUS $\vec{O}$