Note: In ch. 3 all matrices are SQUARE.

3.1 Defn: $detA = \Sigma \pm a_{1j_1} a_{2j_2} ... a_{nj_n}$

2×2 short-cut: $det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}$

3×3 short-cut: $det \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$

Note there is no short-cut for $n \times n$ matrices when $n > 3$.

REQUIRED

Definition of Determinant using cofactor expansion

must learn

Defn: A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.

Defn: Let $A = (a_{ij})$ by an $n \times n$ square matrix. The determinant of A is

1.) If $n = 1$, $detA = a_{11}$.

2.) If $n > 1$, $detA = \Sigma_{k=1}^{n} (-1)^{1+k} a_{1k} detA_{1k}$

$$= a_{11} detA_{11} - a_{12} detA_{12} + ... + (-1)^{1+n} a_{1n} detA_{1n}$$

Note the above definition is an inductive or recursive definition.

$$\begin{pmatrix} + & - & + & + \\ - & + & - & - \\ & & & + \\ + & - & + & - \\ & & & + \\ & & & - \end{pmatrix}$$

$(-1)^{2+3}$

$(-1)^{3+3}$

Thm: Let $A = (a_{ij})$ by an $n \times n$ square matrix, $n > 1$.
Then expanding along row i,

row column

$$detA = \Sigma_{k=1}^{n} (-1)^{i+k} (a_{ik}) detA_{ik}.$$

Or expanding along column j,

$$detA = \Sigma_{k=1}^{n} (-1)^{k+j} a_{kj} detA_{kj}.$$

Defn: $detA_{ij}$ is the i, j-minor of A.

$(-1)^{i+j} detA_{ij}$ is the i, j-cofactor of A.

A_{ij} remove row i column j

~~3.2: Properties of Determinants~~

can also do column ops

Thm: If $A \xrightarrow{R_i \to cR_i} B$, then $detB = c(detA)$.

Warning note: $det(cA) = c^n detA$.

Thm: If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $detB = -(detA)$.

Thm: If $A \xrightarrow{R_i + cR_j \to R_i} B$, then $detB = detA$.

use row ops to create determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow{R_2 + hR_1 \to R_1} \begin{vmatrix} c+ha & d+hb \\ c & d \end{vmatrix}$$

$= \cancel{cd} + had - \cancel{cd} - hcb$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix}$$

1. Expand along row 1
2. " " " 3
3. " " column 2

} shorter method

$$\begin{vmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix}$$

$$+1\begin{vmatrix} 5 & 6 \\ 0 & 8 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 0 \end{vmatrix}$$

$$1(5\cdot 8 - 0\cdot 6) - 2(4\cdot 8 - 7\cdot 6) + 3(4\cdot 0 - 5\cdot 7)$$

$$40 - 2(-10) + 3(-35)$$

$$60 - 105 = -45$$

Expand along row 3

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix}$$

$$\begin{array}{l} + \leftarrow (-1)^{1+1} \\ - \leftarrow (-1)^{2+1} \\ + \; (-1)^{3+1} \; - \; + \end{array}$$

$(-1)^{\text{row + column}}$

$$+7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \underbrace{- 0\begin{vmatrix} \sim \\ \sim \end{vmatrix}}_{=0} + 8\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$7(2\cdot 6 - 5\cdot 3) - 0 + 8(5\cdot 1 - 2\cdot 4)$$

$$= 7(-3) + 8(-3) = 15(-3) = -45$$

Expand along column 2

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix} = \quad \begin{pmatrix} + & - \\ & + \\ & - \end{pmatrix}$$

$$-2\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} - 0\begin{vmatrix} \sim \\ \sim \end{vmatrix}$$

$$= -2(32-42) + 5(8-21)$$

$$= -2(-10) + 5(-13) =$$

$$20 \mp 65 = -45$$

$$\det A_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix}$$

remove
row 2 & column 2

$$A = \begin{vmatrix} 1 & \cancel{2} & 3 \\ \cancel{4} & \cancel{8} & \cancel{6} \\ 7 & & 8 \end{vmatrix} \qquad \begin{pmatrix} + & & \\ - & + & \end{pmatrix}$$

Cofactor $\quad -\begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\Big\updownarrow R_1 \leftrightarrow R_2$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc)$$

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \longrightarrow \begin{matrix} g & h & i \\ a & b & c \\ d & e & f \end{matrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{matrix} d & e & f \\ a & b & c \\ g & h & i \end{matrix}$$

$R_1 \leftrightarrow R_3$

Do only one row switch at a time (or be careful)

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = (-1)^2 \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$\det\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Longrightarrow \det (-1)^2 \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix}$$

$$\xrightarrow{R_1 \Leftrightarrow R_2} -\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} \xleftrightarrow{R_1 \Leftrightarrow R_3}$$

even # of row switches $R_i \Leftrightarrow R_j$ does not change determinant

but an odd # does

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow{R_1 \to kR_1} \begin{vmatrix} ka & kb \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad-bc)$$

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{1}{k}\begin{vmatrix} ka & kb \\ c & d \end{vmatrix}}$$

$$\begin{vmatrix} a & b \\ c^{+ka} & d^{+kb} \end{vmatrix} \xrightarrow{\boxed{1R_2 + kR_1 \to 1R_2}} \begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = a(d+kb) - b(c+ka)$$

$$= ad + \cancel{kab} - bc - \cancel{kba}$$

$$= ad - bc$$

Does not change determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7^{-7} & 0^{-14} & 8^{-21} \end{vmatrix} \xrightarrow{R_3 - 7R_1 \to R_3} \begin{vmatrix} 1 & 2 & 3 \\ 4^{-4} & 5^{-8} & 6^{-12} \\ 0 & -14 & -13 \end{vmatrix}$$

$$\downarrow \; R_2 - 4R_1 \to R_2$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -14 & -13 \end{vmatrix} = 1\left|\;\right| - 0\left|\;\right| + 0\left|\;\right|$$

$$1 \begin{vmatrix} -3 & -6 \\ -14 & -13 \end{vmatrix} = 39 - 84$$

$$= -45$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 1 \cdot 5 \cdot 8 \cdot 10$$

$$= 400$$

SHORT CUT

long method

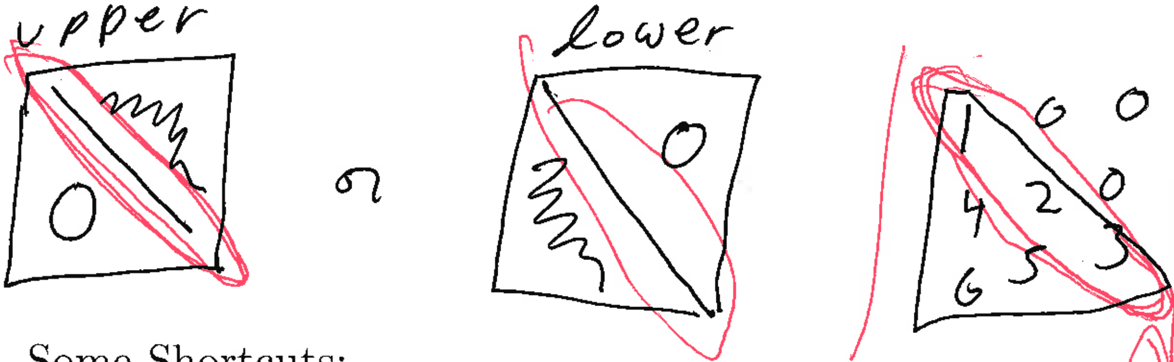
LONG METHOD

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 5 & 6 & 7 \\ 0 & 8 & 9 \\ 0 & 0 & 10 \end{vmatrix} - 0 + 0 - 0$$

$$= 1 \left(5 \begin{vmatrix} 8 & 9 \\ 0 & 10 \end{vmatrix} - 0 + 0 \right)$$

$$= 1 \cdot 5 \cdot 8 \cdot 10 = 400$$



$= 1 \cdot 2 \cdot 3 = 6$

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $detA = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Cor: $det(I_n) = 1$. $\quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Thm: If a square matrix has a row or column containing all zeros, its determinant is zero. $\quad 0 - 0 + 0 - 0 = 0$

Thm: If some row (column) of a square matrix A is a scalar multiple of another row (column), then $\boxed{detA = 0}$. $\quad \begin{matrix} a & b \\ ka & kb \end{matrix}$

row op to create a row of 0's

$$\boxed{\text{Thm: A square matrix is invertible if and only if } detA \neq 0.}$$

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $detA \neq 0$.

$A^{-1}Ax = A^{-1}b \Rightarrow X = A^{-1}b$

unique sol'n

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\quad \shortparallel$

$\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Thm: $\boxed{detAB = (detA)(detB)}$.

$\boxed{\text{Cor: } detA^{-1} = \frac{1}{detA}}$.

$\boxed{det(A + B) \neq detA + detB}$. $\Leftarrow$ don't $det(A+B)$

Thm: $detA^T = detA$.

$\frac{1}{det A} = det\, I = det(AA^{-1}) = \cancel{det A}\, det(A^{-1})$

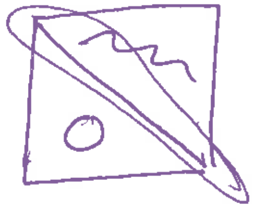
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad det A$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 5 & 8 \end{vmatrix} \xrightarrow[R_2 - 3R_1]{} \begin{vmatrix} 0 & 0 & 0 \end{vmatrix}$$

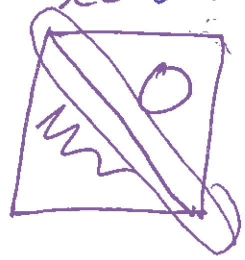
$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 5 & 8 \end{vmatrix} = 0$$

look for matrices where det OBVIOUSLY $= 0$

upper



lower



$$\begin{vmatrix} 1 & 0 & 0 \\ \star & 2 & 0 \\ \triangle & \star & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 = 6$$

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $detA = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Cor: $det(I_n) = 1$. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Thm: If a square matrix has a row or column containing all zeros, its determinant is zero.

expand along this one $\Longrightarrow 0$

Thm: If some row (column) of a square matrix A is a scalar multiple of another row (column), then $detA = 0$.

$R_i - kR_j \longrightarrow$ row of zero's

Thm: A square matrix is invertible if and only if $detA \neq 0$.

Thm: Let A be a square matrix. Then the linear system $Ax = b$ has a unique solution for every b if and only if $detA \neq 0$.

$A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$ ← unique sol'n

Thm: $detAB = (detA)(detB)$.

Cor: $detA^{-1} = \frac{1}{detA}$.

$det(A + B) \neq detA + detB$.

Thm: $detA^T = detA$.

Row ops won't change whether or not det = 0

$$\frac{1}{det} = \frac{det\, I}{det} = \frac{det(A A^{-1})}{det\, A} = \frac{det\, A \; det(A^{-1})}{det\, A}$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{vmatrix} \overset{\text{Shortcut}}{=} (1)(6)(10)(13)(15)$$

$$= (78)(150) \sim$$

$$\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 10 & 11 & 12 \\ 0 & 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 0 & 15 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 6 & 7 & 8 & 9 \\ 0 & 10 & 11 & 12 \\ 0 & 0 & 13 & 14 \\ 0 & 0 & 0 & 15 \end{vmatrix} - 0 + 0 - 0 + 0$$

$$= 1 \cdot \left(6 \begin{vmatrix} 10 & 11 & 12 \\ 0 & 13 & 14 \\ 0 & 0 & 15 \end{vmatrix} - 0 + 0 - 0 \right)$$

$$= 1 \cdot 6 \cdot \left(10 \begin{vmatrix} 13 & 14 \\ 0 & 15 \end{vmatrix} - 0 + 0 \right)$$

$$(1)(6)(10)(13)(15)$$

Note Row ops won't change whether or not det $= 0$

They ~~will~~ can change determinant

But nonzero det will stay nonzero

$c(-\det)$ $\quad c \neq 0$

Proof of thm $detAB = (detA)(detB)$:

Lemma 1:
Let M be a square matrix, and let E be an elementary matrix of the same order. Then $det(EM) = (detE)(detM)$.

Lemma 2: Let M be a square matrix, and let $E_1, E_2, ..., E_k$ be elementary matrices of the same order as M. Then $det(E_1 E_2 ... E_k M) = (detE_1)(detE_2)...(detE_k)(detM)$.

Lemma 3:
Let $E_1, E_2, ..., E_k$ be elementary matrices of the same order. Then $det(E_1 E_2 ... E_k) = (detE_1)(detE_2)...(detE_k)$.

$$\begin{vmatrix} 2 & 1 & 3 & 2 \\ 2^{-2} & 1^{-1} & -1^{-3} & 1^{-2} \\ 0 & 3 & -2 & 1 \\ 4^{-4} & 1^{-2} & 2^{-6} & 2^{-4} \end{vmatrix}$$

$$1R_2 - R_1 \rightarrow 1R_2$$
$$1R_4 - 2R_1 \rightarrow 1R_4$$

$$\begin{vmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & -4 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & -1 & -4 & -2 \end{vmatrix}$$

$$\boxed{R_2 \leftrightarrow R_4}$$

$$-\begin{vmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 3^{+3} & -2^{-12} & 1^{-6} \\ 0 & 0 & -4 & -1 \end{vmatrix} \xrightarrow[R_3 + 3R_2]{} -\begin{vmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & -14^{+20} & -5 \\ 0 & 0 & -4 & -1 \end{vmatrix}$$

$$= 2(-1)\begin{vmatrix} -14 & -5 \\ -4 & -1 \end{vmatrix}$$

$$= +2\,(14 - 20) = -12$$

$$\xrightarrow{R_3 - 5R_4} \; - \begin{vmatrix} 2 & 1 & 3 & 2 \\ 0 & -1 & -4 & -2 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & -4 & -1 \end{vmatrix}$$

$$= -\left(2(-1)\begin{vmatrix} 6 & 0 \\ -4 & -1 \end{vmatrix}\right)$$

$$= 2\,(-6) = -12$$

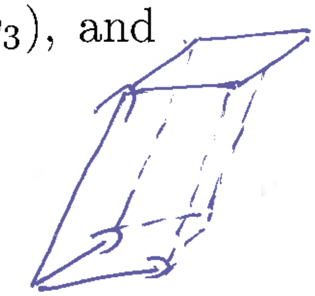
Area and Volume

a.) The area of the parallelogram in 2-space determined by the vectors (u_1, u_2) and (v_1, v_2)

$$= \left| det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$

b.) The volume of the parallelepiped in 3-space determined by the vectors (u_1, u_2, u_3), (v_1, v_2, v_3), and (w_1, w_2, w_3)

$$= \left| det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$



Example: Find the area of the parallelogram determined by the vectors (1, 2) and (3, 4).

Example: Find the area of the parallelepiped determined by vectors (1, 4, 5), (2, 10, 0), & (3, 0, 6)