= span{ }

v_1 & v_2 are soln to $A\vec{x} = \vec{0}$ (homo) $\Rightarrow$ Any linear combi of solns is also a soln to $A\vec{x} = \vec{0}$ (homo)

Suppose $A\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} = \mathbf{0}$, then $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0}$$

NOTE: Nullspace of A = span$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.8 Subspaces of R^n. = Vector space

Long definition emphasizing important points:

Defn: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if and only if the following three conditions are satisfied:

i.) $\mathbf{0}$ is in W,

ii.) if $\mathbf{v_1}, \mathbf{v_2}$ in W, then $\mathbf{v_1} + \mathbf{v_2}$ in W,

iii.) if $\mathbf{v}$ in W, then $c\mathbf{v}$ in W for any scalar c.

closed under linear combination

A subspace is a vector space

Short definition: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if $\mathbf{v_1}, \mathbf{v_2}$ in W implies $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ in W,

span

Note that if S is a subspace, then

if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ in S, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$ is in S.

$0\mathbf{v} = \mathbf{0}$ is in S.

Defn: Let S be a subspace of R^k. A set T is a **basis** for S if

i.) T is linearly independent and — but not overly large don't want extra vectors

ii.) T spans S. — large enough ~~spa~~ for span

Subspace of $\mathbb{R}^n$ = Vector space

is a set closed under linear combinations

I.e S is an finite dimensional subspace

$\iff S = \text{span}\{v_1, \dots, v_n\}$

0-dim subspace

$\{\vec{0}\} = \text{span}\{\vec{0}\}$

not l.i

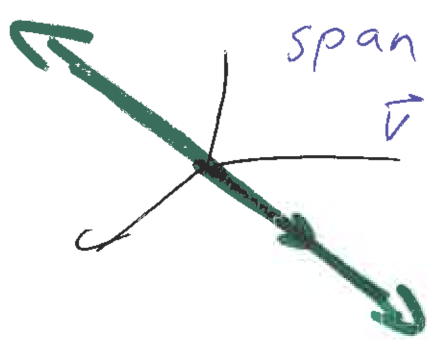
Note $\vec{0}$ is always in S

$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_n$ in S

1-dim subspace

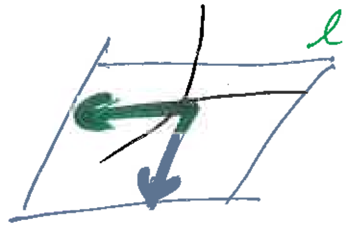
span$\{v\}$

$\vec{v} \neq 0$



2-dim subspace

span$\{\underbrace{v, w}_{\text{l.i}}\}$



2d plane

etc

hyperplane

span$\{v_1, \dots v_n$

Ex: Nullspace

Solution set to $A\vec{x} = \vec{0}$

$\vec{x} = x_i \vec{v} + x_j \vec{w}$ etc

where x_i, x_j are free variables

Null space = Span $\{\vec{v}, \vec{w}\}$

Dim of Nullspace = Nullity = # of free variables

Ex: Col space of $A = [\vec{a}_1 \dots \vec{a}_n]$

Span $\{\vec{a}_1, \dots, \vec{a}_n\}$

simplify answer

take only pivot colums of

RANK = Dim of Col space = # of pivots of A

Basis for S is smallest set of vectors that span S

The simplified answer

2.9: Basis and Dimension

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if

i.) $\mathcal{T}$ is linearly independent and ← NOT TOO LARGE (SIMPLIFIED) remove extra vectors

ii.) $\mathcal{T}$ spans S. ← LARGE ENOUGH

Examples

a.) $\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \}$ is a basis for $span\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \}$ ✓ ← GOLDILOCKS APPROVED

span & li

b.) $\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}} \}$ is NOT a basis for $span\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \}$

TOO LARGE

not l.i.

c.) $\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \}$ is NOT a basis for $span\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \}$

TOO SMALL

Does not span

Defn: A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. Otherwise, V is **infinite dimensional**. $\{1, t, t^2, t^3, \dots\}$

Thm: All basis for a finite-dimensional vector space have the same number of elements.

Defn: $dim(V)$ = the **dimension** of a finite-dimensional vector space V = the number of vectors in any basis for S. If $dim(V) = n$, then V is said to be n-dimensional.

Let $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$.

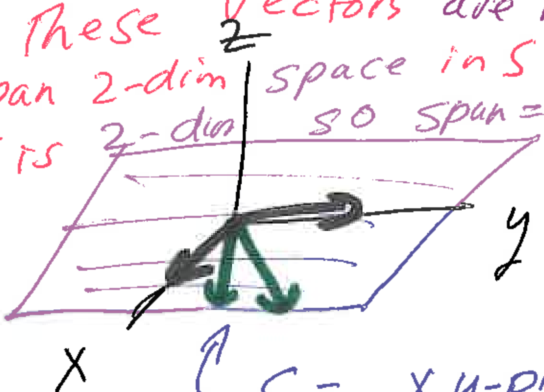
$\boxed{S \subset \mathbb{R}^3}$

Basis for S	Not basis for S
$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ Does not span
$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} \right\}$ Not l. i.
$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

These vectors are in S

Span 2-dim space in S

S is 2-dim so span $= S$



$S =$ xy-plane

Is lin indep

But $\underbrace{\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{Not } S} \neq S$

It is a basis for $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

rank A = Rank of a matrix A = dimension of Col A
= number of pivot columns of A.

nullity of A = dimension of Nul A = number of free variables.

Basis theorem: Let H be a p-dimensional subspace of R^n.

i.)If $H = \text{span}\{w_1, ..., w_p\}$, then $\{w_1, ..., w_p\}$ is a basis for H.

ii.) If $v_1, ..., v_p$ are linearly independent vectors in H,
then $\{v_1, ..., v_p\}$ is a basis for H.

Rank(A) + nullity(A) = Number of columns of A.

# of pivot + # free variables = # of columns of A

Ex. 1) Suppose A is a 5X7 matrix.

If Rank(A) = 4, then nullity(A) = $7 - 4 = 3$

$A\mathbf{x} = \mathbf{0}$ has ∞ solutions. ↑ 3 free variables

$A\mathbf{x} = \mathbf{b}$ has none or ∞ solutions.

If Rank(A) = 5, then nullity(A) = $7 - 5 = 2$

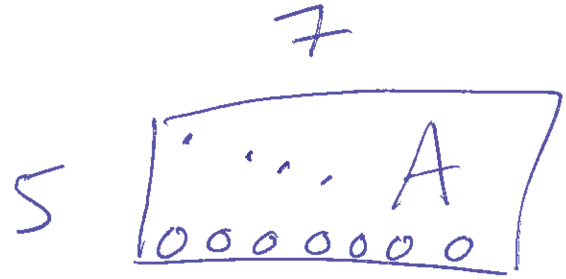
$A\mathbf{x} = \mathbf{0}$ has ∞ solutions. since homog & free variables

$A\mathbf{x} = \mathbf{b}$ has ∞ solutions.

If Rank(A) = 5, the column space of A = $\mathbb{R}^5$

pivot in each row of coef matrix + free variable

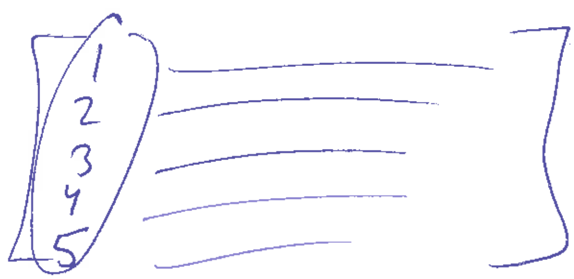
no row of all 0's in coef matrix



4 pivots $\Rightarrow$ 7 - 4 free variable

RANK = 4 $\Rightarrow$ Nullity = 3

Col A is a 4-dimensional subspace of $\mathbb{R}^{\boxed{5}}$



Null A is a 3-dim subspace of $\mathbb{R}^{\boxed{7}}$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$\boxed{5} \times 7 \qquad 7 \times \boxed{1} \qquad \boxed{5} \times \boxed{1}$

EX:

$A \quad 5 \times 7$

$\text{Rank } A = 5$

Col A is a 5-dim subspace of $\mathbb{R}^{\boxed{5}}$

$\Rightarrow \text{col } A = \mathbb{R}^5$

Thm 8′: If A is a $\boxed{\text{SQUARE } n}\times n$ matrix, then the following are equivalent.

a.) A is invertible.

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries
[I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row).

d.) The column vectors of A are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any b.

g.) $Ax = b$ has a unique sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n.
[every vector in R^n can be written as a linear combination of the columns of A].

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

m.) A^T is invertible.

n.) A is expressible as a product of elementary matrices.

SQUARE

$n \times n$ matrix

o.) The column vectors of A form a basis for R^n.
[every vector in R^n can be written uniquely as a linear combination of the columns of A]. { lin indep

p.) Col $A = R^n$.

q.) dim Col $A = n$.

r.) rank of $A = n$.

n pivots

pivot in each row

SQUARE

s.) Nul $A = \{\mathbf{0}\}$,

t.) dim Nul $A = 0$.

u.) A has nullity 0.

n pivots

pivot in each column $\Rightarrow$ no free variable

Rank(A) + nullity(A) = Number of columns of A.

Ex. 2) Suppose A is a $9X4$ matrix.

If Rank(A) = 4, then nullity(A) = $4-4=0$

no free variable

$A\mathbf{x} = \mathbf{0}$ has Unique solutions.

$A\mathbf{x} = \mathbf{b}$ has at most one solutions.

none or exactly one

If Rank(A) = 3, then nullity(A) = $4-3=1$

free variable

$A\mathbf{x} = \mathbf{0}$ has ∞ solutions.

$A\mathbf{x} = \mathbf{b}$ has none or ∞ solutions.

rows of 0's

$A \quad 9 \times 4$

$$\begin{bmatrix} \; & \; & \; & \; \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_9 \end{bmatrix}$$

$$9 \times \underline{4} \quad \underline{4} \times 1 = 9 \times 1$$

$$\text{Nul } A \subset \mathbb{R}^4$$

$$\text{Col } A \subset \mathbb{R}^9$$

Note: In ch. 3 all matrices are SQUARE.

3.1 Defn: $detA = \Sigma \pm a_{1j_1}a_{2j_2}...a_{nj_n}$

2 × 2 short-cut: $det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$

3 × 3 short-cut: $det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

Note there is no short-cut for $n \times n$ matrices when $n > 3$.

REQUIRED

Definition of Determinant using cofactor expansion ⟸ must learn

Defn: A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.

Defn: Let $A = (a_{ij})$ by an $n \times n$ square matrix. The determinant of A is

1.) If $n = 1$, $detA = a_{11}$.

2.) If $n > 1$, $detA = \Sigma_{k=1}^{n}(-1)^{1+k}a_{1k}detA_{1k}$

$$= a_{11}detA_{11} - a_{12}detA_{12} + ... + (-1)^{1+n}a_{1n}detA_{1n}$$

Note the above definition is an $\boxed{\text{inductive}}$ or $\boxed{\text{recursive}}$ definition.

$$\left|\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right.$$

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

This method only works for 3x3

optional

$$\det\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = (1)(4) - (2)(3)$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cofactor expansion

$$= ad - bc$$

$$(-1)^{1+1} ad + (-1)^{1+2} bc$$

$$\begin{vmatrix}1&2&3\\4&5&6\\7&0&8\end{vmatrix}$$

① Expand along row 1

② " " " 3

③ " " column 2

} shorter method

$$\begin{vmatrix}1&2&3\\4&5&6\\7&0&8\end{vmatrix}$$

$$+1\begin{vmatrix}5&6\\0&8\end{vmatrix}-2\begin{vmatrix}4&6\\7&8\end{vmatrix}+3\begin{vmatrix}4&5\\7&0\end{vmatrix}$$

$$1(5\cdot 8-0\cdot 6)-2(4\cdot 8-7\cdot 6)+3(4\cdot 0-5\cdot 7)$$

$$40-2(-10)+3(-35)$$

$$60-105=-45$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= 0 + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \cancel{0\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}} + 8\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$= 0$

$$= 7(12-15) + 8(5-8)$$

$$= 7(-3) + 8(-3) = 15(-3) = -45$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 8 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix} \quad (-1)^{1+2}$$

$$-2\begin{vmatrix} 4 & 6 \\ 7 & 8 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 7 & 8 \end{vmatrix} - \cancel{0\begin{vmatrix} & \\ & \end{vmatrix}}$$

$= 0$

$$= -2(32-42) + 5(8-21)$$

$$= -2(-10) + 5(-13) = 20-65 = -45$$

$(-1)^{1+1}$ $(-1)^{1+}$

$(-1)^{2+1}$

$(-1)^{3+1}$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Thm: Let $A = (a_{ij})$ by an $n \times n$ square matrix, $n > 1$. Then expanding along row i, row column

$$detA = \Sigma_{k=1}^{n}(-1)^{i+k}a_{ik}detA_{ik}.$$

Or expanding along column j,

$$detA = \Sigma_{k=1}^{n}(-1)^{k+j}a_{kj}detA_{kj}.$$

Defn: $detA_{ij}$ is the i, j-minor of A. remove row i column j

$(-1)^{i+j}detA_{ij}$ is the i, j-cofactor of A.

3.2: Properties of Determinants

Thm: If $A \xrightarrow{R_i \to cR_i} B$, then $detB = c(detA)$.

Warning note: $det(cA) = c^n detA$.

Thm: If $A \xrightarrow{R_i \leftrightarrow R_j} B$, then $detB = -(detA)$.

Thm: If $A \xrightarrow{1R_i + cR_j \to 1R_i} B$, then $detB = detA$.

Do row ops to get more 0's

$1R_i + cR_j \to 1R_i$

$R_i + cR_j \to 1R_j$ ← affects determ

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7^{-7} & 0^{-14} & 8^{-21} \end{vmatrix} \xrightarrow{1R_3 - 7R_1 \to 1R_3} \begin{vmatrix} 1 & 2 & 3 \\ 4^{-4} & 5^{-8} & 6^{-12} \\ 0 & -14 & -13 \end{vmatrix}$$

$$1R_2 - 4R_1 \to 1R_2$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -14 & -13 \end{vmatrix} = +1 \begin{vmatrix} -3 & -6 \\ -14 & -13 \end{vmatrix} - 0 + 0$$

$$= 1(+39 - 84) = -45$$