homogeneous = 0

homogeneous

2.8 Subspaces of R^n.

Example: The **nullspace of** A is the solution set of $A\mathbf{x} = \mathbf{0}$. = nul(A) = null(A)

Coef   solve

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ⇐ EF

∞ # of sol'ns

$$\text{Nullspace of } A = \text{Solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

$$= \text{solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$\downarrow R_1 - 2R_2$

$$= \text{solution space of } \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$ REF

Solve

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array}$$

free

$$\begin{bmatrix} x_1 = \\ x_2 = \\ x_3 = \\ x_4 = \end{bmatrix} \begin{bmatrix} -3 \cdot x_3 & - 4x_4 \\ 0 & \\ 1x_3 & \\ & 1x_4 \end{bmatrix}$$

4 variables
⇒ solution set for $Ax=0$ will be a subspace of $\mathbb{R}^4$

$$\vec{x} = \begin{bmatrix} -3x_3 - 4x_4 \\ 0 \\ 1x_3 \\ 1x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 0 \\ 1x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ 1x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

free variables: $x_3 = x_3$ $\quad x_4 = x_4$

Basis

$$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

lin indep

Nullspace of A

$= $ solution set of $Ax = 0$

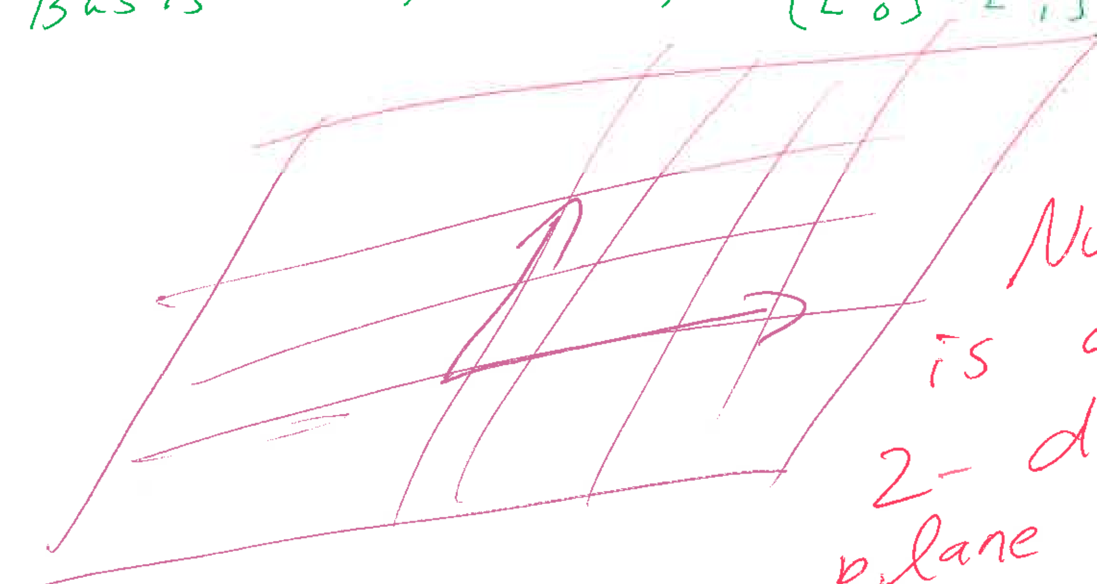
$$= \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \;\middle|\; \begin{matrix} x_3 \in \mathbb{R} \\ x_4 \in \mathbb{R} \end{matrix} \right\}$$

set of all linear combinations of 2 vectors

$$= \text{span} \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$x_3 = x_3$ $\qquad$ $x_4 = x_4$

Basis for Null $(A) = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$



Nullspace is a 2-dim plane in $\mathbb{R}^4$

Note $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4$ since it has 4 coordinates

= span { }

v_1 & v_2 are soln to $A\vec{x}=\vec{0}$ (homo)

Any linear combi of solns is also a soln to $A\vec{x}=\vec{0}$ (homo)

Suppose $A\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} = \mathbf{0}$, then $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = \vec{0}+\vec{0} = \vec{0}$$

NOTE: Nullspace of A = span$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.8 Subspaces of R^n. = Vector space

Long definition emphasizing important points:

Defn: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if and only if the following three conditions are satisfied:

i.) $\mathbf{0}$ is in W,

ii.) if $\mathbf{v_1}, \mathbf{v_2}$ in W, then $\mathbf{v_1} + \mathbf{v_2}$ in W,

iii.) if $\mathbf{v}$ in W, then $c\mathbf{v}$ in W for any scalar c.

closed under linear combination

A subspace is a vector space

Short definition: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if $\mathbf{v_1}, \mathbf{v_2}$ in W implies $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ in W, ≈ span

---

Note that if S is a subspace, then

if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ in S, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$ is in S.

$0\mathbf{v} = \mathbf{0}$ is in S.

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if

i.) $\mathcal{T}$ is linearly independent and ← but not overly large don't want extra vectors

ii.) $\mathcal{T}$ spans S.

large enough ~~to spa~~ for span

EX: set of negative #'s is closed under +

neg + neg = neg

They are not closed under ×

$(-2)(-3) = +6$

not neg

Example of closed under ___

Any space that can be written as a span of vectors will be a vector space (subspace)

Examples: Nullspace and Column Space.

Let $A = [\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}]$, a $k \times n$ matrix.

Defn: The **column space of** $A = span\{\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}\}$ = col(A)

Thm: The column space of A is a subspace of R^k

Note: Suppose B is row equivalent to A, then the column space of B need not be the same as the column space of A.

$$A = \begin{bmatrix} 1 & 2 & \cancel{3} & \cancel{4} \\ 2 & 5 & \cancel{6} & \cancel{8} \\ 3 & 7 & \cancel{9} & \cancel{12} \\ 1 & 2 & \cancel{3} & \cancel{4} \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4}$$

throw out free variable columns to get linear independence

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

EF

The column space of $A = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix}}, \cancel{\begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix}} \right\}$

not simplified

Don't need all 4 columns to specify the span

$\boxed{\infty \text{ \# of vectors}}$

pivot columns

$= span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$.

simplified

Thus a basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$.

$\boxed{\text{2 vectors}}$

since 2 pivots

Only need pivot columns

Col space of A

col A = span of col of A

REF ↓ To find basis

throw out free variable columns

pivot columns of A

span ⇆ l.i

Nul A = soln set $Ax = 0$

Solve $Ax = 0$

REF

solution = linear comb of vectors

1 vector for each free variable

Solve to find the vectors that span Nul A

Get lin indep for free

pivot columns for simplified answer

Note we took the leading entry columns in the ORIGINAL matrix.

Why are we so interested in the column space?

b in col A $\Longleftrightarrow$ $A\vec{x}=\vec{b}$ has a soln

Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 + \cancel{\begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix}} x_3 + \cancel{\begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix}} x_4 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a sol'n?

free variable columns (extra) choose $x_3=0$ $x_4=0$

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Is $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$ = column space of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix}$?

$\begin{bmatrix} 9 \\ 22 \\ 31 \\ 9 \end{bmatrix}$ is in col A

pivot

Example 1: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \\ 9 \end{bmatrix}$ have a sol'n? YES

Example 2: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 8 \\ 4 \end{bmatrix}$ have a sol'n? NO!

$\begin{bmatrix} 3 \\ 7 \\ 8 \\ 4 \end{bmatrix}$ is not in col A

Long method for determining IF there is a solution:

$$\left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & 9 & 3 \\ 2 & 5 & 8 & 7 & 22 & 7 \\ 3 & 7 & 12 & 8 & 31 & 8 \\ 1 & 2 & 5 & 4 & 9 & 4 \end{array}\right] \to \left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right]$$

pivots

Shorter method for determining IF there is a solution WHEN you know a basis for the column space:

$$\left[\begin{array}{cc|cc} 1 & 2 & 9 & 3 \\ 2-2 & 5-4 & 22-18 & 7-6 \\ 3-3 & 7-6 & 31-27 & 8-9 \\ 1-1 & 2-2 & 9-9 & 4-3 \end{array}\right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1}} \left[\begin{array}{cc|cc} 1 & 2 & 9 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

$0 \neq 1$ no soln

$R_3 - R_2$

$$\left[\begin{array}{cc|cc} 1 & 2 & 9 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

soln

not pivot

A basis for S consists of the fewest vectors that describe S

simplified answer

2.9: Basis and Dimension

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a $\boxed{\textbf{basis}}$ for S if

NOT TOO LARGE (SIMPLIFY)

i.) $\mathcal{T}$ is linearly independent and

ii.) $\mathcal{T}$ spans S.

large enough to span S

Examples

a.) $\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}\}$ is a basis for $span\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}\}$ ✓ GOLDILOCKS APPROVED

l.i & spans ✓ just right

b.) $\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}, \cancel{\begin{bmatrix}4\\6\\0\end{bmatrix}}\}$ is NOT a basis for $span\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}\}$ NOT LIN INDEP

TOO LARGE LIN DEP FREE VAR

c.) $\{\begin{bmatrix}1\\2\\0\end{bmatrix}\}$ is NOT a basis for $span\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\4\\0\end{bmatrix}\}$ DOES NOT SPAN

TOO SMALL

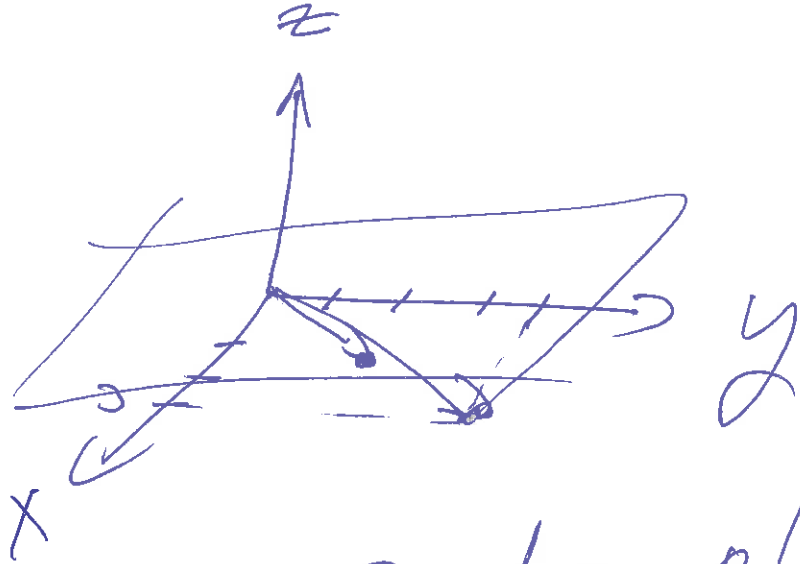
Defn: A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. Otherwise, V is **infinite dimensional**. $\{1, t, t^2, t^3, \ldots\}$

$\boxed{\text{Thm:}}$ All basis for a finite-dimensional vector space have the same number of elements.

Defn: $dim(V)$ = the **dimension** of a finite-dimensional vector space V = the number of vectors in any basis for S. If $dim(V) = n$, then V is said to be n-dimensional.

A basis for S is the smallest set of vectors needed to describe set S

$$\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right\}$$



span 2-dim plane

in $\mathbb{R}^3$.

$$\begin{array}{cc} & \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{matrix} \\ \text{5 rows} & \underbrace{\begin{bmatrix} 1 & \ast & \ast & \ast & \ast & \ast & \ast \\ 2 & \ast & & & & & \\ 3 & \ast & & & & & \\ 4 & \ast & & & & & \\ 5 & \ast & \ast & \ast & \ast & \ast & \ast \end{bmatrix}}_{7} \end{array}$$

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ is a vector in $\mathbb{R}^5$

$A\vec{x} = 0$ $\qquad$ A 5×7

$(5) \times \underline{7} \quad (\underline{7} \times \boxed{1}) = (5) \times \boxed{1}$

case $\quad$ RANK $A = 4$.

dim of Nul A = Nullity = 3

Nullspace is a 3-dim hyperplane living in $\mathbb{R}^7$

$$5\begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$7$$

$$\text{Nul } A \subset \mathbb{R}^7$$

Thm 8′: If A is a $\boxed{\text{SQUARE}}$ $n \times n$ matrix, then the following are equivalent.

a.) A is invertible.

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries
[I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row).

d.) The column vectors of A are linearly independent.

e.) $Ax = 0$ has only the trivial solution.

f.) $Ax = b$ has at most one sol'n for any b.

g.) $Ax = b$ has a unique sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n.
[every vector in R^n can be written as a linear combination of the columns of A].

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

m.) A^T is invertible.

n.) A is expressible as a product of elementary matrices.

$n \times n$ matrix A

o.) The column vectors of A form a basis for R^n. [every vector in R^n can be written uniquely as a linear combination of the columns of A].

span

p.) Col $A = R^n$.

q.) dim Col $A = n$.

r.) rank of $A = n$.

pivot in every row

l.i

# of rows $= n =$ # of columns

s.) Nul $A = \{\mathbf{0}\}$,

t.) dim Nul $A = 0$.

u.) A has nullity 0.

solution set to $Ax=0$

pivot in every column

$A\vec{x}=\vec{0} \Rightarrow \vec{x}=\vec{0}$ is the only soln

0-dim

$n - n = 0$

n col − n pivots

Rank(A) + nullity(A) = Number of columns of A.

Ex. 2) Suppose A is a $9X4$ matrix. 4 col

If Rank(A) = 4, then nullity(A) = $4-4=0$

4 col. all pivots No f.v.

$A\mathbf{x} = \mathbf{0}$ has unique solutions. $x=0$

$A\mathbf{x} = \mathbf{b}$ has ______________ solutions.

If Rank(A) = 3, then nullity(A) =

$A\mathbf{x} = \mathbf{0}$ has ______________ solutions.

$A\mathbf{x} = \mathbf{b}$ has ______________ solutions.