SQUARE COEF MATRIX A

n pivots in each row $\Longleftrightarrow$ n pivots in each column

Thm 8′: If A is a $\boxed{\text{SQUARE}}$ $n \times n$ matrix, then the following are equivalent.

$A^{-1}Ax = A^{-1}b$

$x = A^{-1}b$

unique soln

a.) $\boxed{A \text{ is invertible}}$

b.) The row-reduced echelon form of A is I_n, the identity matrix.

c.) An echelon form of A has n leading entries [I.e., every column of an echelon form of A is a leading entry column – no free variables]. (A square => A has leading entry in every column if and only if A has leading entry in every row). no row of all 0's

d.) The column vectors of A are linearly independent. pivot in each column

no f. v.

e.) $Ax = 0$ has only the trivial solution.

SQUARE

f.) $Ax = b$ has at most one sol'n for any b.

SQUARE

g.) $Ax = b$ has a $\boxed{\text{unique}}$ sol'n for any b.

h.) $Ax = b$ is consistent for every $n \times 1$ matrix b.

i.) $Ax = b$ has at least one sol'n for any b.

j.) The column vectors of A span R^n. [every vector in R^n can be written as a linear combination of the columns of A].

k.) There is a square matrix C such that $CA = I$.

l.) There is a square matrix D such that $AD = I$.

m.) A^T is invertible. $\Leftarrow$ same # of pivots as A

~~n.) A is expressible as a product of elementary matrices.~~

COEF MATRIX SQUARE MATRIX

3x3

$n = 3$ pivots in both cases if invertible

Different set of equations

$CA = I$

$AD = I$

Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix}$.

Long method: $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 6 & 7 & 9 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} =$

$$\begin{bmatrix} 2x_{11} + 3x_{21} + 4x_{31} & 2x_{12} + 3x_{22} + 4x_{32} & 2x_{13} + 3x_{23} + 4x_{33} \\ 4x_{11} + 5x_{21} + 6x_{31} & 4x_{12} + 5x_{22} + 6x_{32} & 4x_{13} + 5x_{23} + f x_{33} \\ 6x_{11} + 7x_{21} + 9x_{31} & 6x_{12} + 7x_{22} + 9x_{32} & 6x_{13} + 7x_{23} + 9x_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So solve,

$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 0 \end{array}\right]$ for x_{11}, x_{21}, x_{31}. $\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in span of col of A

$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 1 \\ 6 & 7 & 9 & 0 \end{array}\right]$ for x_{12}, x_{22}, x_{32}. $\Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ''

$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 4 & 5 & 6 & 0 \\ 6 & 7 & 9 & 1 \end{array}\right]$ for x_{13}, x_{23}, x_{33}. $\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ''

$\Downarrow$

if span $= \mathbb{R}^3$

A^{-1} exists

Or shorter method, solve

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 4 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 6 & 7 & 9 & 0 & 0 & 1 \end{array}\right]$$

Solving 9 systems of equations

A^{-1} exists $\Longleftrightarrow$ $[A \mid I]$ has a sol'n

Thm: Let A be a square matrix. If there exists a square matrix B such that $AB = I$, then $BA = I$ and thus $B = A^{-1}$ $\Rightarrow$

Thm: If A is invertible, then its inverse is unique.

Proof: Suppose $AB = I$ and $CA = I$. Then, $B = IB = CAB = CI = C$.

Defn: $A^0 = I$, and if n is a positive integer $A^n = AA \cdots A$ and $A^{-n} = A^{-1}A^{-1} \cdots A^{-1}$.

Thm: If r, s integers, $A^r A^s = A^{r+s}$, $(A^r)^s = A^{rs}$

Thm: If A^{-1} and B^{-1}exist, then

i.) $\boxed{AB \text{ is invertible and } (AB)^{-1} = B^{-1}A^{-1}}$

ii.) A^{-1} is invertible and $(A^{-1})^{-1} = A$ obvious

$AA^{-1} = I$

iii.) A^r is invertible and $(A^r)^{-1} = (A^{-1})^r$ where r is any integer

iv.) For any nonzero scalar k, kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

v.) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Augmented

$$n \; \underbrace{\left[\, \underbrace{A}_{n} \;\middle|\; b \,\right]}_{n+1}$$

$$n \times (n+1)$$

If A has **n** pivots

then do NOT have

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & \star \end{array}\right]$$

pivot

Thus you have

$$\left[\begin{array}{cccc|c} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \end{array}\right]$$

always have a sol'n

NO PIVOT

homogeneous = 0.

homogeneous

2.8 Subspaces of R^n.

Example: The **nullspace of** A is the solution set of $A\mathbf{x} = \mathbf{0}$. = nul(A) = null(A)

coef   solve

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \xrightarrow{R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

EF

∞ # of sol'ns

$$\text{Nullspace of } A = \text{Solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

$$= \text{solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$\downarrow R_1 - 2R_2$

$$= \text{solution space of } \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

RREF

Solve $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$\begin{array}{cccc} x_1 & x_2 & \boxed{x_3} & \boxed{x_4} \end{array}$ free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 - 4x_4 \\ 0 \\ 1x_3 \\ 1x_4 \end{bmatrix}$$

4 variables solution

$\Rightarrow$ solution set for $Ax=0$ will be a subspace of $\mathbb{R}^4$

$$\vec{x} = \begin{bmatrix} -3x_3 - 4x_4 \\ 0 \\ 1x_3 \\ 1x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 0 \\ 1x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ 1x_4 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$$

Nullspace of A

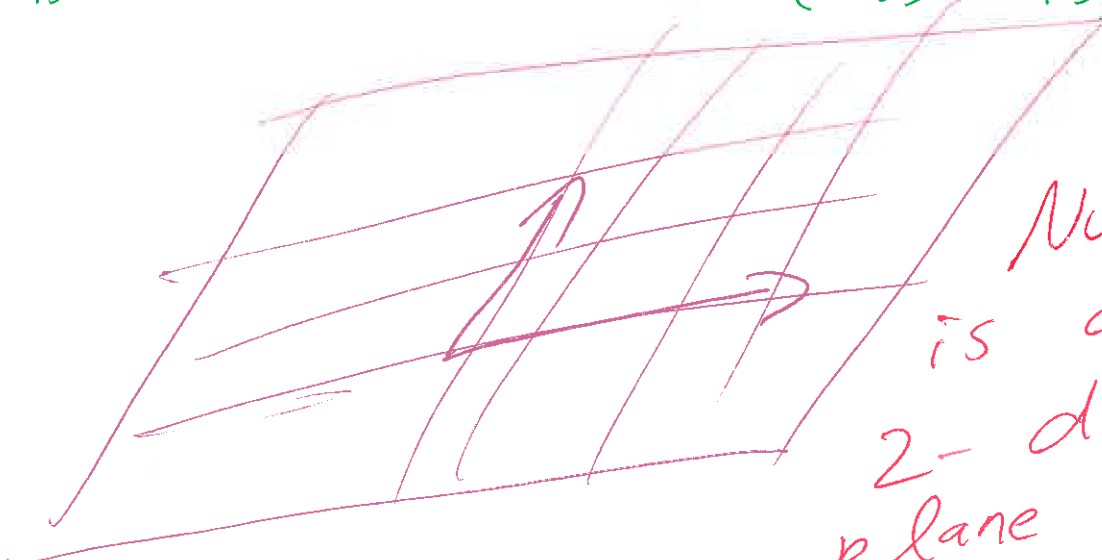
$= $ solution set of $Ax = 0$

$$= \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4 \;\middle|\; \begin{matrix} x_3 \in \mathbb{R} \\ x_4 \in \mathbb{R} \end{matrix} \right\}$$

set of all linear combination of 2 vectors

$$= \text{span} \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Basis for Null(A) $= \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$



Nullspace is a 2-dim plane in $\mathbb{R}^4$

Note $\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4$ since it has 4 coordinates

$= \text{span}\{\quad\}$

v_1 & v_2 are soln to $A\vec{x}=\vec{0}$ (homo) $\Rightarrow$ Any linear combi of solns is also a soln to $A\vec{x}=\vec{0}$ (homo

Suppose $A\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} = \mathbf{0}$, then $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$

$$A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = \vec{0}+\vec{0} = \vec{0}$$

NOTE: Nullspace of A = span$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

2.8 Subspaces of R^n. = Vector space

Long definition emphasizing important points:

Defn: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if and only if the following three conditions are satisfied:

i.) $\mathbf{0}$ is in W,

ii.) if $\mathbf{v_1}, \mathbf{v_2}$ in W, then $\mathbf{v_1} + \mathbf{v_2}$ in W,

iii.) if $\mathbf{v}$ in W, then $c\mathbf{v}$ in W for any scalar c.

closed under linear combination

A subspace is a vector space

Short definition: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if $\mathbf{v_1}, \mathbf{v_2}$ in W implies $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ in W,

& span

Note that if S is a subspace, then

if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ in S, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$ is in S.

$0\mathbf{v} = \mathbf{0}$ is in S.

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if

i.) $\mathcal{T}$ is linearly independent and ← but not overly large don't want extra vectors

ii.) $\mathcal{T}$ spans S.

large enough ~~to spa~~ for span

EX: set of negative #'s is closed under +

neg + neg = neg

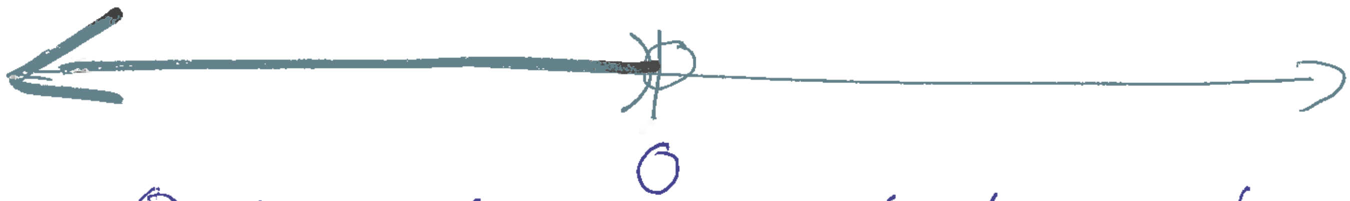
They are not closed under $\times$

$(-2)(-3) = +6$

not neg

Example of closed under__

Exs of a set that is not a subspace



0 is not in the set of negative #'s so the set neg #'s is not a subspace of $\mathbb{R}^1$

The soln set to a non-homog eqn

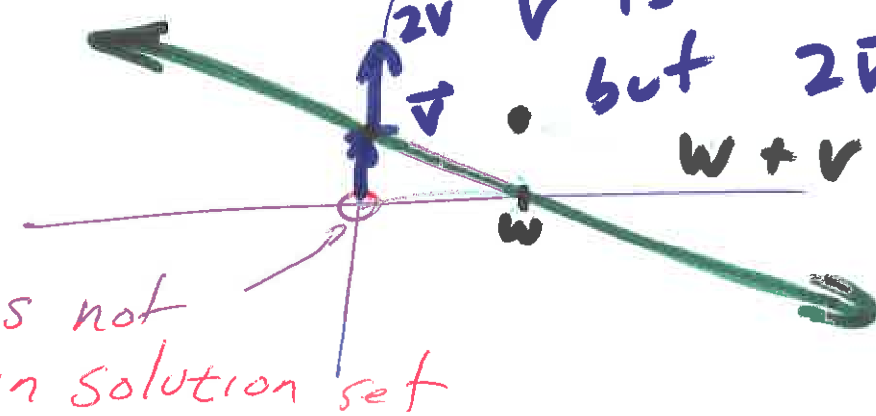
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Not a subspace

$x + 2y = 3 \Rightarrow y = -\frac{1}{2}x + 3$

$\vec{v}$ is in soln set but $2\vec{v}$ is not

w + v is not a sol'n



$\vec{0}$ is not in solution set

EX:

Not subspaces

Solution set of

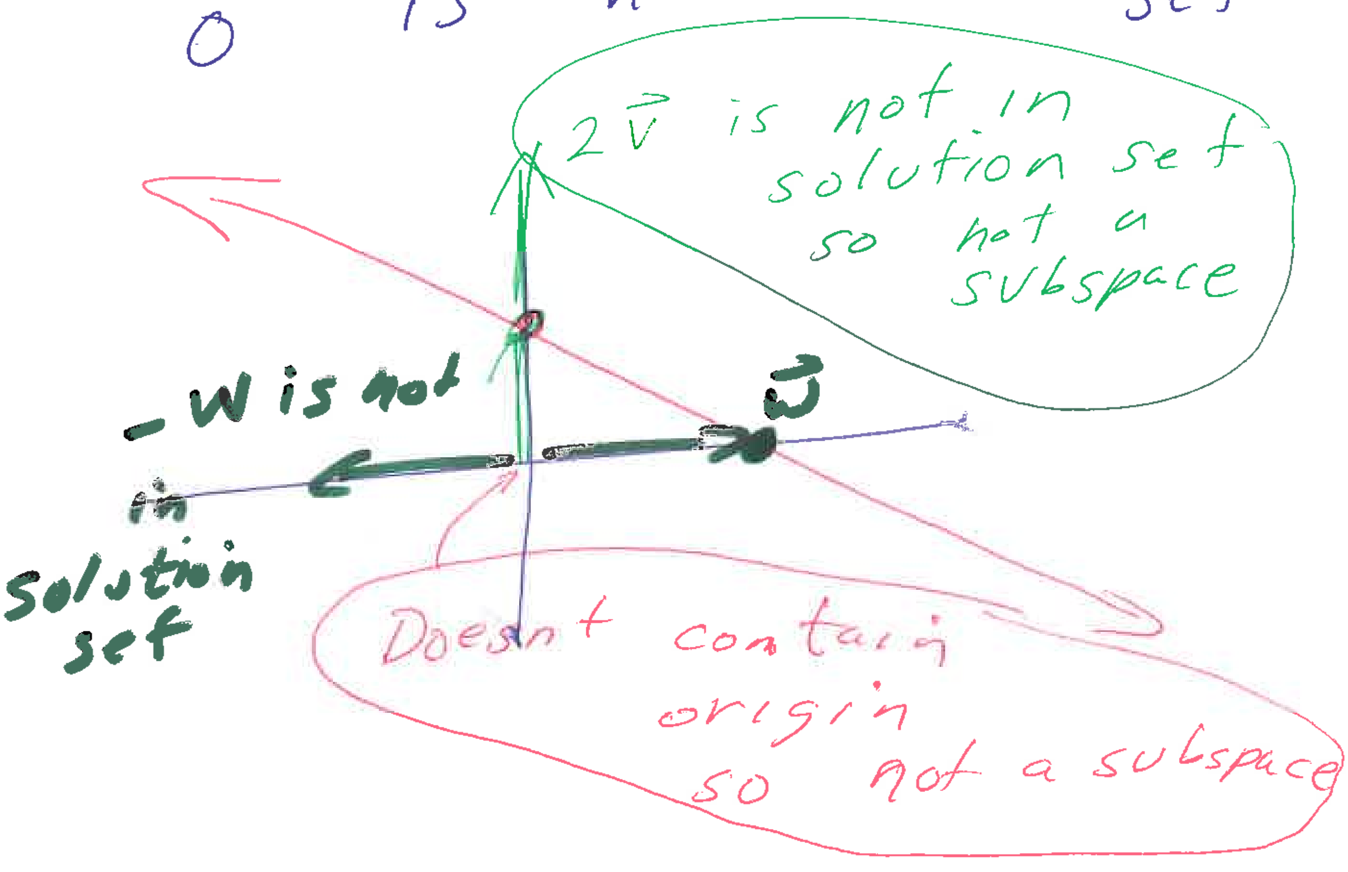
$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2y = 3 \Rightarrow y = -\frac{1}{2}x + \frac{3}{2}$

Not a subspace

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

so $\vec{0}$ is not in solution set



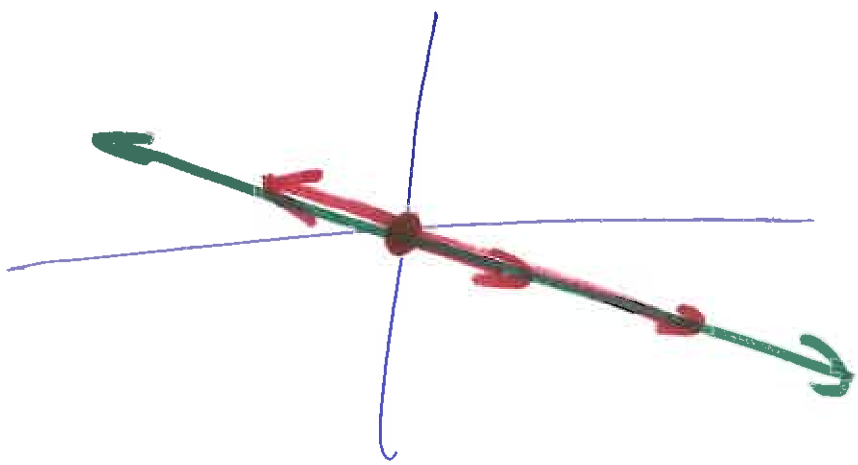
Subspace = Nul C

$$C = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Solution set to

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$y = -\frac{1}{2} x$



is a subspace of $\mathbb{R}^2$

Any space that can be written as a span of vectors will be a vector space (subspace)

Examples: Nullspace and Column Space.

Let $A = [\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}]$, a $k \times n$ matrix.

Defn: The **column space of** $A = span\{\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}\}$ = col(A)

Thm: The column space of A is a subspace of R^k

Note: Suppose B is row equivalent to A, then the column space of B need not be the same as the column space of A.

$$A = \begin{bmatrix} 1 & 2 & \cancel{3} & \cancel{4} \\ 2 & 5 & \cancel{6} & \cancel{8} \\ 3 & 7 & \cancel{9} & \cancel{12} \\ 1 & 2 & \cancel{3} & \cancel{4} \end{bmatrix} \xrightarrow{R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4}$$

throw out free variable columns to get linear independence

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The column space of $A = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix}}, \cancel{\begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix}} \right\}$ ⇐ not simplified

pivot columns

$= \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$.

Don't need all 4 columns to specify the span

Thus a basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$.

only need pivot columns

Note we took the leading entry columns in the ORIGINAL matrix.

Why are we so interested in the column space?

Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix} x_3 + \begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix} x_4 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a sol'n?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Is $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\} =$ column space of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix}$?

pivot

Example 1: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \\ 9 \end{bmatrix}$ have a sol'n?

Example 2: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 8 \\ 4 \end{bmatrix}$ have a sol'n?

Long method for determining IF there is a solution:

$$\left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & 9 & 3 \\ 2 & 5 & 8 & 7 & 22 & 7 \\ 3 & 7 & 12 & 8 & 31 & 8 \\ 1 & 2 & 5 & 4 & 9 & 4 \end{array}\right] \to \left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right]$$

Shorter method for determining IF there is a solution WHEN you know a basis for the column space: