12:30

To solve a system of equations:

i.) Create augmented matrix.

ii.) Put matrix into EF.

iii.) Put into REF.

iv.) Solve.

To determine # of sol'ns

to find sol'n

Case 1: If pivot in last column of augmented matrix.
Then system of equations has **no solution**.

$0x + 0y + 0z = 5$

$0 = 5$ a contradiction

Case 2: If no pivot in last column of augmented matrix:

a.) No free variables implies **unique solution**.

b.) Free variables imply an **infinite number of solutions**
Solve for pivot column variables in terms of free variables.

Stop as soon as you know sol'n

12:30

1.3 Vectors in R^m

Defn: $\mathbf{u} = (u_1, ..., u_m)$, $\mathbf{v} = (v_1, ..., v_m)$ are **vectors in** $\mathbf{R^m}$.

Defn: $u_1, ..., u_m$ are the **components** of $\mathbf{u}$.

Defn: $\mathbf{u} = \mathbf{v}$ if and only if $u_i = v_i$ for all i.

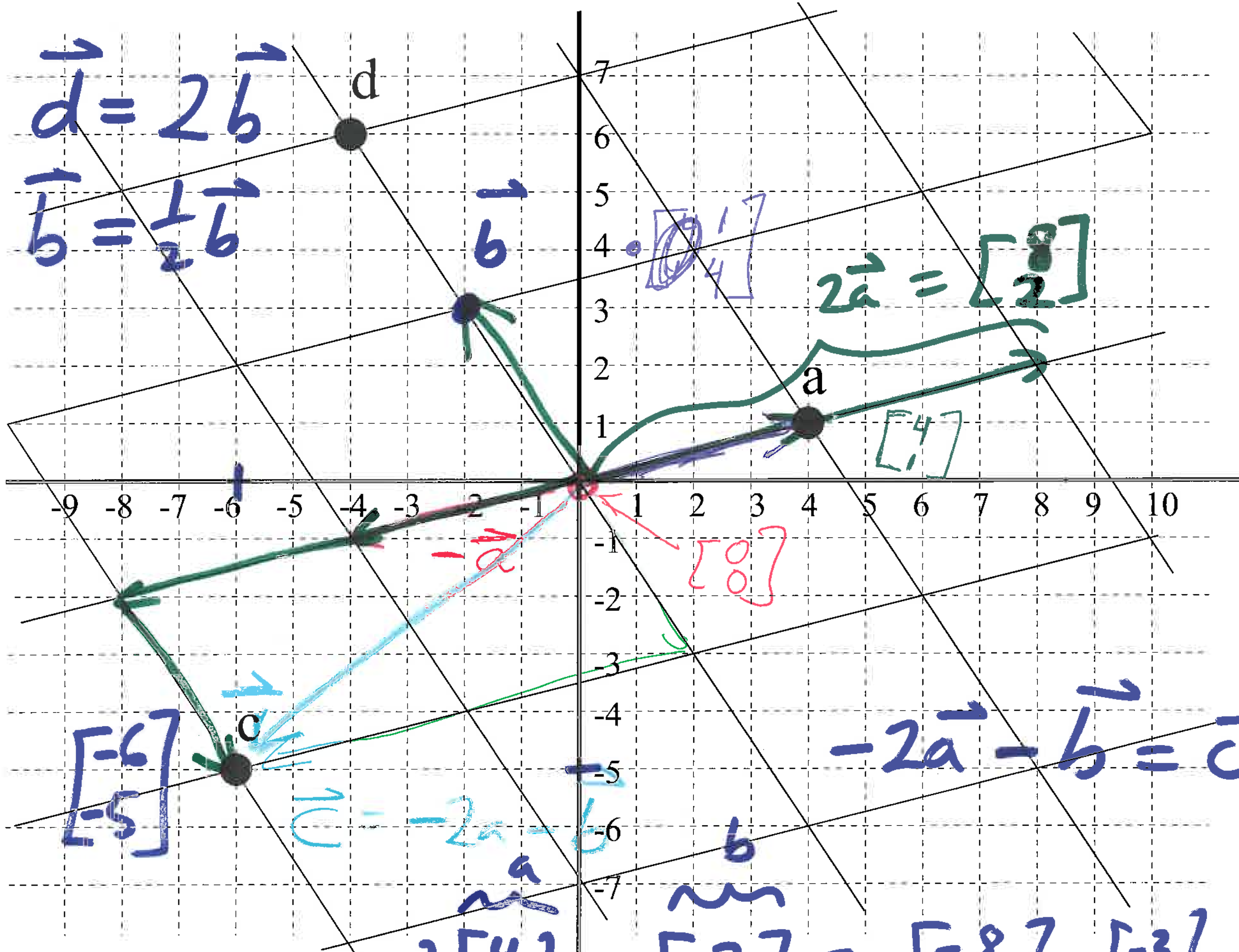
Defn: The **zero vector** in $\mathbf{R^m}$ is the m-vector $\mathbf{0} = (0, 0, ..., 0)$.

Vector Addition

Defn: The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = (u_1 + v_1, ..., u_m + v_m)$.

Defn: The **negative** of $\mathbf{u}$ is the vector $-\mathbf{u} = (-u_1, ..., -u_m)$

Defn: The **difference** between $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, ..., u_m - v_m)$.



$\vec{d} = 2\vec{b}$

$\vec{b} = \frac{1}{2}\vec{b}$

$2\vec{a} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 8 \end{bmatrix}$

$-\vec{a}$

$\vec{c} = -2\vec{a} - \vec{b}$

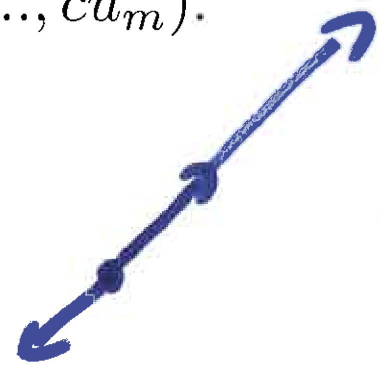
$-2\vec{a} - \vec{b} = \vec{c}$

$\begin{bmatrix} -6 \\ -5 \end{bmatrix}$

$-2\underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{a} - \underbrace{\begin{bmatrix} -2 \\ 3 \end{bmatrix}}_{b} = \begin{bmatrix} -8 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$

Defn: In this class a **scalar**, c, is a real number.

Defn: The **scalar multiple** of $\mathbf{u}$ by c is the vector $c\mathbf{u} = (cu_1, ..., cu_m)$.



Thm: The vectors, $\mathbf{u}$ and $\mathbf{v}$, are collinear iff there exists a scalar c such that $\mathbf{v} = c\mathbf{u}$. In this case

a.) if $c > 0$, $\mathbf{u}$ and $c\mathbf{u}$ have the same direction.

b.) If $c < 0$, $\mathbf{u}$ and $c\mathbf{u}$ have opposite directions.

Defn: The *length (norm, magnitude)* of $\mathbf{u}$ is its distance from $\mathbf{0}$ and is denoted by

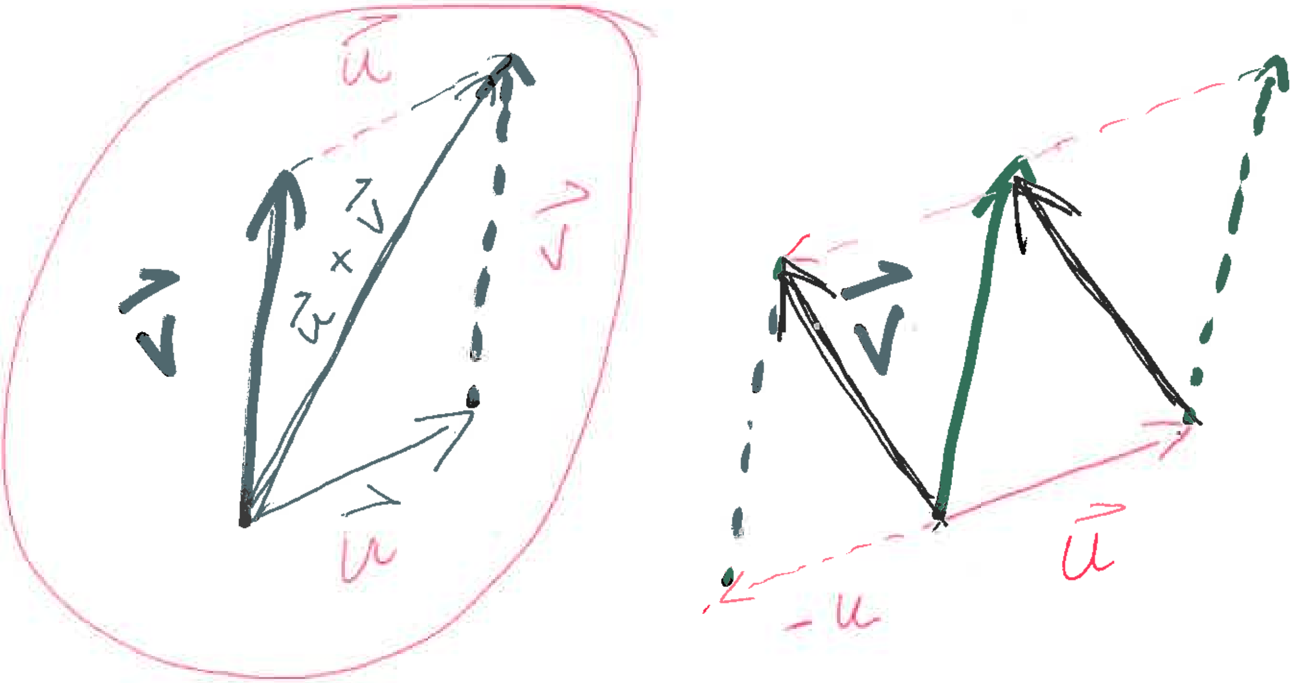
$$||\mathbf{u}|| = d(\mathbf{0}, \mathbf{u}) = \sqrt{u_1^2 + u_2^2 + ... + u_m^2}.$$

Two vectors are equivalent if they have the same direction and length.

Parallelogram rule:

Addition: the directed line segment starting at $\mathbf{u}$ and ending at $\mathbf{u}+\mathbf{v}$ is equivalent to $\mathbf{v}$

Subtraction: the directed line segment starting at $\mathbf{u}$ and ending at $\mathbf{v}$ is equivalent to $\mathbf{v}-\mathbf{u}$



Note $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \neq \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$

However, we will sometimes abuse notation.

Thm 3.2.1 (or thm 4.1.1 p163)

a.) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

b.) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

c.) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

d.) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

e.) $(cd)\mathbf{u} = c(d\mathbf{u})$

f.) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

g.) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

h.) $1\mathbf{u} = \mathbf{u}$

Sometimes we will write the vector $\mathbf{x}$ as a row vector: $(x_1, \ldots, x_n)$.

Other times we will write the vector $\mathbf{x}$ as a column vector: $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$

augmented coef

constants

Section 1.4

$$\begin{bmatrix} \underbrace{a_{11} \quad a_{12}}_{} & \Big| & b_1 \\ a_{21} \quad a_{22} & & b_2 \end{bmatrix}$$

augmented matrix form

↓

EF

↓

REF

Solve

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

↓

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

↓

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

↓

1.3

$$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

vector format

1.4

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{coeff}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\text{constant}}$$

Matrix format

$$\begin{bmatrix} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Solve:

$$\begin{aligned} x_1 \quad + \qquad\qquad\quad 6x_3 &= 7 \\ x_2 \quad + \quad 8x_3 &= 9 \end{aligned}$$

Solve:

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \end{bmatrix}$$

1.3 Vectors in R^m

Defn: The vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ if there exist scalars $c_1, \ldots, c_n$ such that

$$\mathbf{w} = c_1\vec{\mathbf{v_1}} + c_2\vec{\mathbf{v_2}} + \ldots + c_n\vec{\mathbf{v_n}}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$[redacted]

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} c_1 + \begin{bmatrix} 4 \\ 8 \end{bmatrix} c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ is a linear comb of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$\Longleftrightarrow$ there exists sol'n for c_1 & c_2

$$\begin{bmatrix} 9c_1 \\ 7c_1 \end{bmatrix} + \begin{bmatrix} 4c_2 \\ 8c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 9c_1 + 4c_2 \\ 7c_1 + 8c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7 & 8 & -5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7-\frac{7}{9}(9) & 8-\frac{7}{9}(4) & -5-\frac{7}{9}(3) \end{array}\right]$$

$$\Big\downarrow$$

$$R_2 - \frac{7}{9}R_1 \rightarrow R_2$$

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 0\left(\frac{9}{44}\right) & \frac{44}{9}\left(\frac{9}{44}\right) & -\frac{66}{9}\left(\frac{9}{44}\right) \end{array}\right]$$

EF
so'n exists

$$\Big\downarrow \; R_2\left(\frac{9}{44}\right)$$

$$\left[\begin{array}{cc|c} 9 & 4-4 & 3-4\left(-3/2\right) \\ 0 & 1 & -3/2 \end{array}\right]$$

$$\Big\downarrow \; R_1 - 4R_2$$

$$\left[\begin{array}{cc|c} 9/9 & 0 & 9/9 \\ 0 & 1 & -3/2 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = -3/2 \end{array}$$

Write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as lin comb

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} c_1 + \begin{bmatrix} 4 \\ 8 \end{bmatrix} c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

Ans: $\boxed{\begin{bmatrix} 9 \\ 7 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}}$

Check answer: $\begin{bmatrix} 9 \\ 7 \end{bmatrix} - \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ ✓

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & \frac{44}{9} & -\frac{66}{9} \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 9 & 0 & 9 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

Thus, $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2) \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

If possible, write $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ [redacted]

$$\underline{0} \begin{bmatrix} 9 \\ 7 \end{bmatrix} + \underline{0} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}$ [redacted]

Not possible

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear comb'n of $\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -30 \\ 50 \end{bmatrix}$ [redacted]

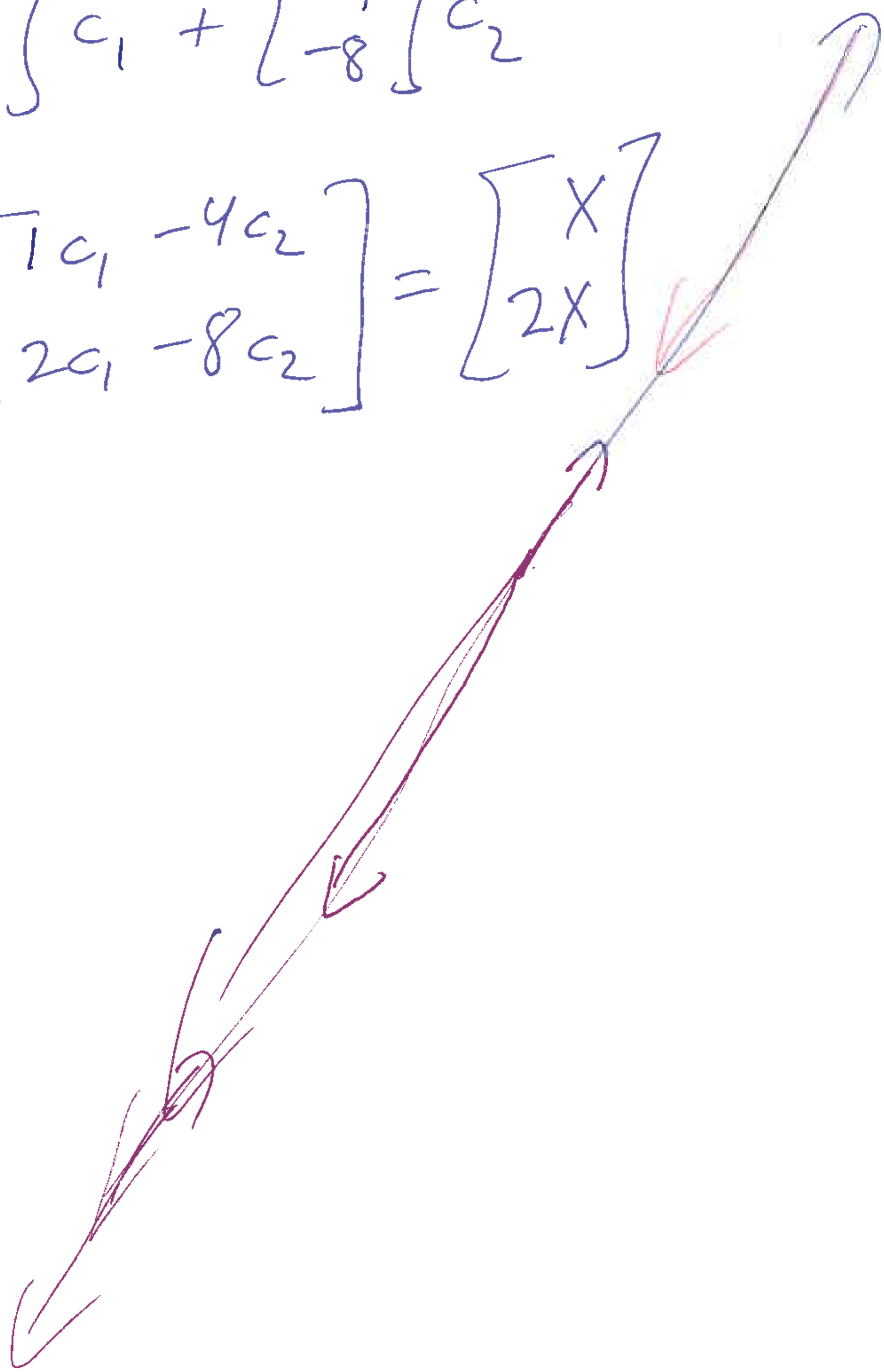
$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 0 \begin{bmatrix} -30 \\ 50 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 2^{-2} & -8^{+8} & -5^{-6} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 0 & -11 \end{array}\right]$$

no sol'n

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1 + \begin{bmatrix} -4 \\ -8 \end{bmatrix} c_2$$

$$\begin{bmatrix} 1c_1 - 4c_2 \\ 2c_1 - 8c_2 \end{bmatrix} = \begin{bmatrix} X \\ 2X \end{bmatrix}$$



Write $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} -30 \\ 50 \end{bmatrix}$

Long method

$$\left[\begin{array}{cc|c} 3 & -30 & 3 \\ -5 & 50 & -5 \end{array}\right]$$

$$\Big\downarrow \begin{matrix} R_1/3 \\ R_2/-5 \end{matrix}$$

$$\begin{bmatrix} 1 & -10 & 1 \\ 1 & -10 & 1 \end{bmatrix}$$

$$\Big\downarrow R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & -10 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$1c_1 - 10c_2 = 1$$

$$c_1 = -10c_2 + 1$$
$$c_2 = c_2$$

∞ # of sol'n

choose 1

$$c_2 = 0 \Rightarrow c_1 = 1 = -10(0) + 1$$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 0 \begin{bmatrix} -30 \\ 50 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$c_2 = 1 \Rightarrow c_1 = -10(1) + 1 = -9$$

$$-9 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + \begin{bmatrix} -30 \\ 50 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

Both answers are correct

choose one