9/22

1.3 $A\vec{x}=\vec{b}$

3x2 matrix

3 rows
2 columns $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ $\mathbb{R}^3$

two vectors in $\mathbb{R}^3$

A has
K rows
n columns

$\text{span}\{\vec{a}_1, \dots \vec{a}_n\} = \mathbb{R}^K$

$\{c_1\vec{a}_1 + c_2\vec{a}_2 + \dots + c_n\vec{a}_n \mid c_i \in \mathbb{R}\} = \mathbb{R}^K$

$\Longleftrightarrow$ $A\vec{x}=\vec{b}$ has at least one sol'n for any $\vec{b}$

$\Longleftrightarrow$ pivot in each row of coef matrix A

1.7 $A\vec{x}=\vec{b}$

columns of A are lin indep

No free variables

$\Longleftrightarrow$ $A\vec{x}=\vec{b}$ has at most one sol'n

$\Longleftrightarrow$ pivot in each column of coef matrix A

1.7 linear independance

$A\,X = \vec{0}$ homog eqn

$\boxed{\text{Coefficient matrix } A}$

unique soln

$\Updownarrow$

no free variables

$\Updownarrow$

Every column of coef matrix is a pivot columns

$\Updownarrow$

Columns of A are lin indep

there exists column which is NOT a pivot COLUMN

∞ # of sol'ns

$\Updownarrow$

free variables

$\Updownarrow$

columns of A lin dependent

lin ind/dep

$A\vec{x} = \vec{b}$, $\vec{b} \neq 0$ non homog

$\boxed{\text{coefficient matrix } A}$

$\boxed{\text{no free variables}}$

$\Downarrow$

unique soln OR no soln

Ex: $\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \end{array}\right]$

unique soln

$\left[\begin{array}{cc|c} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{array}\right]$ ← augmented

$* \neq 0 \Rightarrow$ no soln

$* = 0 \Rightarrow$ unique soln

$\Big\}$ at most one soln $\Leftrightarrow$ columns lin indep

$\boxed{\text{free variables}}$

$\Downarrow$

∞ # of sol'ns OR no sol'n

$\left[\begin{array}{ccc|c} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & * \end{array}\right]$

f.v

$* \neq 0 \Rightarrow$ no sol'n

$* = 0 \Rightarrow \infty$ # of soln

columns lin dep

Section 1.3 : span

$$A\bar{x} = \vec{b}$$

Coefficient matrix A

Does span of columns of $A = \mathbb{R}^K$

$K = \#$ of rows

pivot in each **row** of coef matrix A

$\Updownarrow$

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \end{array}\right]$$

pivots

At least one sol'n for all $\vec{b}$ in $\mathbb{R}^K$

there exists a **row** in coef matrix A that does **NOT** contain a **pivot**

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & * \end{array}\right]$$

row of zeros

(ie no pivot in this row for coef matrix

$\Rightarrow$ No sol'n is a possible answer

Ch 5 Review Questions:

augmented; coef; 2 pivots; coef matrix in EF; fv fv

$$C = \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix}}_{\text{coef}} \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

row of all zeros in EF of coeff matrix

0.) Does $C\mathbf{x} = \mathbf{b}$ have at most one solution for all $\mathbf{b}$?

f.v $\Rightarrow$ ∞ # of soln or no soln NO

1.) Does $C\mathbf{x} = \mathbf{0}$ have exactly one solution?

fv $\Rightarrow$ no

2.) In an echelon form of C, is there a leading entry in every COLUMN?

f.v. $\Rightarrow$ NO

3.) Is $\mathbf{0}$ the only solution to $C\mathbf{x} = \mathbf{0}$?

fv $\Rightarrow$ NO

4.) Are the columns of C linearly independent?

fv $\Rightarrow$ NO

5.) Are none of the columns of C a linear comb'n of the other columns of C? NO

Any free variable column can be written as a lin comb of other columns

6.) Are none of the columns of C in the span of the other columns of C? NO

fv $\Rightarrow$ NO

all the same question just worded differently

Are there free variables?

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

0.) Does $C\mathbf{x} = \mathbf{b}$ have more than one solution for some $\mathbf{b}$?

e.g $\vec{b} = \vec{0}$ f.v

1.) Does $C\mathbf{x} = \mathbf{0}$ have an infinite number of solutions?

f.v.

2.) Are there free variables in the solution to $C\mathbf{x} = \mathbf{0}$?

3.) Does $C\mathbf{x} = \mathbf{0}$ have a non-zero solution?

f.v. e.g. (1, 1, -1, 0)

4.) Are the columns of C linearly dependent?

yes

5.) Is one of the columns of C a linear comb'n of the other columns of C?

6.) Is one of the columns of C in the span of the other columns of C?

SAME QUESTION WORDED DIFF

YES

f.v.

If possible, write one of the columns of C as a linear combination of the other columns of C:

lin comb format

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$$

1rst col   2nd col   3rd col

non zero soln (1, 1, -1, 0)

soln to $Cx=0$

$$1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} - 0\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

EF of C

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

no pivot in coef matrix

1.) Does $C\mathbf{x} = \mathbf{b}$ have at $\boxed{\text{least one solution}}$ for all $\mathbf{b}$?

2.) Does $C\mathbf{x} = \mathbf{b}$ have a solution for all $\mathbf{b}$?

3.) In an echelon form of C, are there NO rows of all zeros?

4.) In an echelon form of C, is there a leading entry in every ROW?

5.) Can any vector in R^{3} be written as a linear comb'n of the columns of C?

6.) Do the columns of C span R^{3}? dim of span = # of pivots = 2

NO

row of zeros in coef matrix

Ie row w/o pivot

1b.) Find a solution to the equation $C\mathbf{x} = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$.

$\left[\, C \;\middle|\; \begin{matrix} 3 \\ 7 \\ 6 \end{matrix} \,\right] \to$ REF

∞ # of soln or no soln

2b.) Write $\begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ as a linear combination of the columns of C.

$\left[\, C \;\middle|\; \begin{matrix} 3 \\ 7 \\ 6 \end{matrix} \,\right] \to$ REF

choose a soln to lin. comb if one exists

all similar questions

3b.) Write $3 + 7t + 6t^2$ as a linear combination of $\{\underbrace{1 + t + 2t^2}_{}, \underbrace{2 + 4t + 4t^2}_{}, 3 + 5t + 6t^2, 4 + 4t + 4t^3\}$.

later class

$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

1a.) Does $C\mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ have at least one solution? NO

No soln

1b.) Does $C\mathbf{x} = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ have at least one solution? Yes

---

2a.) Is $\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$ a linear combination of the columns of C: NO

No soln

2b.) Is $\begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ a linear combination of the columns of C: Yes

in the span of the columns of C: YES

3a.) Is $4 + 2t$ a linear combination of $\{1 + t + 2t^2,\ 2 + 4t + 4t^2,\ 3 + 5t + 6t^2,\ 4 + 4t + 4t^3\}$? NO

3b.) Is $3 + 7t + 6t^2$ a linear combination of $\{1 + t + 2t^2,\ 2 + 4t + 4t^2,\ 3 + 5t + 6t^2,\ 4 + 4t + 4t^3\}$? YES

later class

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 4 \\ 2 & 4 & 6 & 8 \end{bmatrix} \xrightarrow{R_2 - R_1 \to R_2,\, R_3 - 2R_1 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D$$

$$\left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 4 & 3 \\ 1^{-1} & 4^{-2} & 5^{-3} & 4^{-4} & 2^{-4} & 7^{-3} \\ 2^{-2} & 4^{-4} & 6^{-6} & 8^{-8} & 0^{-8} & 6^{-6} \end{array}\right] \to \left[\begin{array}{cccc|cc} 1 & 2 & 3 & 4 & 4 & 3 \\ 0 & 2 & 2 & 0 & -2 & 4 \\ 0 & 0 & 0 & 0 & -8 & 0 \end{array}\right]$$

coef

EF to determine if possible

no soln

∞ # of soln

$R_1 - R_2$

$R_2/2$

REF

$$\left[\begin{array}{cccc|cc} 1 & 0 & 1 & 4 & \cancel{8} & -1 \\ 0 & 1 & 1 & 0 & \cancel{8} & 2 \\ 0 & 0 & 0 & 0 & \cancel{8} & 0 \end{array}\right]$$

Solve $\quad Cx = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ ⇐ section 1.5 problem

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 1 & 4 & 5 & 4 & 7 \\ 2 & 4 & 6 & 8 & 6 \end{array}\right] \longrightarrow \begin{array}{c} \begin{array}{cccc|c} 0 & -2 & -2 & 0 & 4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 2 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Leftarrow \text{EF}$$

$$\Big\downarrow \; \begin{array}{l} R_1 - R_2 \\ R_2/2 \end{array}$$

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \; \text{REF} \\ \begin{array}{cccc} & & x_3 & x_4 \end{array} \Leftarrow \text{f. vs} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 4x_4 - 1 \\ -x_3 + 2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 - 4x_4 - 1 \\ -x_3 + 2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

parametric vector format

$$= s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Write $\begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$ as a linear comb of cols of C

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 1 & 4 & 5 & 4 & 7 \\ 2 & 4 & 6 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Choose a sol'n

$x_3 = x_4 = 0$

$x_1 + 0 + 0 + (0) = -1$

$x_2 + 0 + 0 = 2$

$x_3 = 0$

$x_4 = 0$

$$-1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 6 \end{bmatrix}$$

4 rows $\qquad$ 6 colmns $\Rightarrow$ fv

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 7 \end{bmatrix} \right\}$ linearly independent? NO

Is $\left\{ \begin{array}{c} \text{fv} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$ linearly independent? NO

Is $\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$ linearly independent? YES

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$ linearly independent? NO

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccc} & & \text{fv} \end{array} \\ \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

[redacted]

EF

10:30

Is $\{9+7t, 4+8t, 3-5t\}$ linearly independent? later class

Thm: Let S be a set of n vectors in R^k where $n > k$. Then S is linearly dependent.

Thm: A set of vectors is linearly dependent if one of the vectors can be written as a linear combination of the other vectors.

A set of vectors is linearly independent if none of the vectors can be written as a linear combination of the other vectors.

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \right\}$ linearly independent? NO

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ linearly independent? YES

Is $\left\{ \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$ linearly independent? Yes