12:30
9/17

A NON-homogeneous system of LINEAR equations

a.) Exactly one solution.

b.) Infinite number of solutions

c.) No solutions

A system of equations is $A\mathbf{x} = \mathbf{b}$ is **homogeneous** if $\mathbf{b} = \mathbf{0}$.

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

$A\vec{x} = 0$

b.) Infinite number of solutions
(including, of course, $\mathbf{x} = \mathbf{0}$).

$A\vec{0} = \vec{0}$

Solve the following systems of equations:

A) $$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

B) $$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

No soln

C) $$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

homog

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

$\downarrow R_2 - 4R_1 \to R_2, \quad R_3 - 7R_1 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & -7 & 0 & -6 \end{array}\right]$$

$\downarrow R_3 - 2R_1 \to R_3$

$$\begin{array}{c} \begin{array}{cccccc} & & & A & B & C \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right] \end{array} \quad \leftarrow \text{EF}$$

No sol'n

$\downarrow$ already know sol'n to system b.

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\downarrow -\frac{1}{3}R_2 \to R_2$

f.v.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 - 2R_2 \to R_1]{} \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

REF

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{row op}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 is free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3$$

homog

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \xrightarrow{EF} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

no sol'n

$0 \neq -6$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3+1 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underbrace{\underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} x_3}_{\text{homog soln}} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{a non homog soln}}}_{\text{non homog sol}}$$

Note that $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ and $A(c\mathbf{x}) = cA\mathbf{x}$

For example,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}(x_1+y_1) & a_{12}(x_2+y_2) \\ a_{21}(x_1+y_1) & a_{22}(x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{11}y_1 & a_{12}x_2 + a_{12}y_2 \\ a_{21}x_1 + a_{21}y_1 & a_{22}x_2 + a_{22}y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 & a_{12}x_2 \\ a_{21}x_1 & a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{11}y_1 & a_{12}y_2 \\ a_{21}y_1 & a_{22}y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$2(x+y) = 2x+2y$

$\cancel{\sqrt{2+2}}$

$\sqrt{2+3} \neq \sqrt{2}+\sqrt{3}$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \left(c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}cx_1 & a_{12}cx_2 \\ a_{21}cx_1 & a_{22}cx_2 \end{bmatrix} = c \begin{bmatrix} a_{11}x_1 & a_{12}x_2 \\ a_{21}x_1 & a_{22}x_2 \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Suppose $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$, and $A\mathbf{p} = \mathbf{b}$

p is a sol'n to non hom Ax=b

u & v are sol'n to homog eqn Ax=0

$\vec{u}$ & $\vec{v}$ solns to homog eqn $A\vec{x} = \vec{0}$

$\Longleftrightarrow A\vec{v} = 0$ & $A\vec{u} = \vec{0}$

Claim: $s\vec{u} + t\vec{v}$ is also a soln to $A\vec{x} = \vec{0}$

pf: $A(s\vec{u} + t\vec{v})$

$= A(s\vec{u}) + A(t\vec{v})$

$= s(A\vec{u}) + t(A\vec{v})$

$= s(\vec{0}) + t(\vec{0}) = \vec{0}$ ✓

$\Longrightarrow s\vec{u} + t\vec{v}$ is a soln to homog $A\vec{x} = \vec{0}$

Any linear combination of sol'ns to a homog eqn will also be a soln

$\vec{u}, \vec{v}$ solns to homog eqn $A\vec{x} = \vec{0}$

$\vec{p}$ a soln to nonhomog eqn $A\vec{x} = \vec{b}$

$\Longleftrightarrow A(s\vec{u} + t\vec{v}) = \vec{0}$

& $A\vec{p} = \vec{b}$

<u>Claim</u>: $(s\vec{u} + t\vec{v}) + (\vec{p})$ is a soln to nonhomog $A\vec{x} = \vec{b}$

Pf: $A(s\vec{u} + t\vec{v} + \vec{p})$

$= A(s\vec{u} + t\vec{v}) + A\vec{p}$

$= \vec{0} + \vec{b} = \vec{b}$ ✓

$\Longrightarrow \boxed{s\vec{u} + t\vec{v} + \vec{p}}$ is a soln to $A\vec{x} = \vec{b}$

homog soln + a non homog soln

REF

Solve: $B\,\mathbf{x} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ where $B \sim$

$$\begin{array}{c} \left[\begin{array}{ccccc|c} \boxed{0} & 1 & 0 & \boxed{8} & 0 & 1 \\ \boxed{0} & 0 & 1 & \boxed{-6} & 0 & 2 \\ \boxed{0} & 0 & 0 & \boxed{0} & 1 & 3 \end{array}\right] \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}x_1\\-8x_4+1\\-6x_4+2\\x_4\\3\end{bmatrix} = \begin{bmatrix}x_1\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-8x_4\\-6x_4\\x_4\\0\end{bmatrix} = \underbrace{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}x_1 + \begin{bmatrix}0\\-8\\-6\\1\\0\end{bmatrix}x_4}_{\text{homog}} + \begin{bmatrix}0\\1\\2\\0\\3\end{bmatrix}$$

homog + A soln to non homo

Solution to non-homog eqn

Only need EF of coef matrix to determine if there are free variables

no free variables in coef matrix

1.7: Linear Independence.

Defn: The set of vectors $\{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ is **linearly independent** if and only if the equation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} = \mathbf{0}$ has only the trivial solution.

Defn: If S is not linearly independent, then it is **linearly dependent**.

free variable of coef matrix

Is $\left\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix}\right\}$ is linearly independent? NO

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix}$$

2 rows
3 columns
$\Rightarrow$ at most 2 pivots

coef matrix

$\Rightarrow$ free variable

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$$

$\Rightarrow$ lin dep

free variable

$$A\vec{x} = \vec{0}$$

$$\underbrace{\left[\begin{array}{ccc|c} 9 & 4 & 3 & 0 \\ 7 & 8 & -5 & 0 \end{array}\right]}_{\text{augmented}}$$

$$\downarrow$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & \boxed{\star} & 0 \\ 0 & 1 & \boxed{\star} & 0 \end{array}\right] \\ \text{f.v} \end{array}$$

$$x_3 =$$

1.7 linear independance

$$A x = \vec{0}$$

homog eqn

$\boxed{\text{Coefficient matrix } A}$

unique sol'n

$\Updownarrow$

no free variables

$\Updownarrow$

Every column of coef matrix is a pivot columns

$\Updownarrow$

Columns of A are lin indep

∞ # of sol'n's

$\Updownarrow$

free variables

there exists column which is NOT a pivot COLUMN

$\Updownarrow$

columns of A lin dependent

lin ind/dep

$A\vec{x} = \vec{b}$, $\vec{b} \neq 0$ nonhomog

$\boxed{\text{coefficient matrix } A}$

no free variables

$\Downarrow$

unique soln OR no soln

Ex:

$$\left[\begin{array}{cc|c} 1 & 0 & \star \\ 0 & 1 & \star \end{array}\right]$$

unique soln

$$\left[\begin{array}{cc|c} 1 & 0 & \star \\ 0 & 1 & \star \\ 0 & 0 & * \end{array}\right] \leftarrow \text{augmented}$$

$* \neq 0 \Rightarrow$ no sol'n

$* = 0 \Rightarrow$ unique sol'n

free variables

$\Downarrow$

∞ # of sol'ns OR no sol'n

$$\left[\begin{array}{ccc|c} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 0 & * \end{array}\right]$$

f.V

$* \neq 0 \Rightarrow$ no sol'n

$* = 0 \Rightarrow \infty$ # of soln

columns lin dep

at most one soln $\Leftrightarrow$ columns lin indep

1.3

$span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ = the set of all linear combinations, $c_1\mathbf{v_1}+c_2\mathbf{v_2}+...+c_n\mathbf{v_n}$, of the vectors in $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ $\blacksquare$

= the hyperplane containing the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ anchored at $\mathbf{b} = \mathbf{0}$

= the hyperplane containing the points $\mathbf{0}, \mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$

Let $A = [\mathbf{a_1}...\mathbf{a_n}]$, where the a_i are k-vectors.

1.3 $\mathbf{b}$ is in $span\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ if and only if $Ax = b$ has at least one solution.

$span\{\mathbf{a_1}, ..., \mathbf{a_n}\} = R^k$ if and only if $Ax = b$ has at least one solution for every $\mathbf{b}$ (leading entry in every row).

Section 1.3 : span

$$A\bar{x} = \vec{b}$$

Coefficient matrix A

Does span of columns of $A = \mathbb{R}^K$
$K = \#$ of rows

pivot in each **row** of coef matrix A

$\Updownarrow$

$$\begin{bmatrix} \star & \star & \star & \bigg| & \star \\ 0 & \star & \star & \bigg| & \star \end{bmatrix}$$

pivots

At least one sol'n for all $\vec{b}$ in $\mathbb{R}^K$

there exists a **row** in coef matrix A that does <u>NOT</u> contain a **pivot**

$$\begin{bmatrix} \star & \star & \star & \bigg| & \star \\ 0 & 0 & 0 & \bigg| & * \end{bmatrix}$$

row of zeros

ie no pivot in this row for coef matrix

$\Rightarrow$ No sol'n is a possible answ

1.3 $A\vec{x}=\vec{b}$

A has
k rows
n columns

$\text{span}\{\vec{a}_1, \dots \vec{a}_n\} = \mathbb{R}^k$

$\Longleftrightarrow$ $A\vec{x}=\vec{b}$ has at least one sol'n for any $\vec{b}$

$\Longleftrightarrow$ pivot in each row of coef matrix A

1.7 $A\vec{x}=\vec{b}$

columns of A are lin indep

$\Longleftrightarrow$ $A\vec{x}=\vec{b}$ has at most one sol'n

$\Longleftrightarrow$ pivot in each column of coef matrix A

10:30

Note only need E.F.

$\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ is linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has exactly one solution (pivot in every column of echelon form of coefficient matrix A).

$\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ is linearly dependent if and only if $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions. (at least one free variable)

Thm: Let $\mathcal{S} = \{\mathbf{a_1}, \mathbf{a_2}, ..., \mathbf{a_n}\}$ be a set of vectors in R^k. Then $\mathcal{S}$ is linearly dependent if and only if the vector equation $c_1\mathbf{a_1} + c_2\mathbf{a_2} + ... + c_n\mathbf{a_n} = \mathbf{0}$ has an infinite number of solutions.

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$ linearly independent? NO

Standard Method 1

$$\begin{bmatrix} 1 & 4 & 5 \\ 2^{-2} & 5^{-8} & 7^{-10} \\ 3^{-3} & 6^{-12} & 9^{-15} \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

~~$R_3 - (R_1 + R_2)$~~

F.V

l.d

Method 2

$$\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}5\\7\\9\end{bmatrix}$$

← note one colum is a linear combinatin of the other columns

$\Leftrightarrow$ lin dep

$\Rightarrow$ lin dep

why?

$$\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\5\\6\end{bmatrix}-\begin{bmatrix}5\\7\\9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{c|c|c}1&4&5\\2&5&7\\3&6&9\end{array}\right]\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

A non-trivial sol'n

$$\begin{bmatrix}1&4&5\\2&5&7\\3&6&9\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

A trivial sol'n

2 sol'ns

$\Rightarrow \infty$ # of solns $\Rightarrow$ free variable $\Rightarrow$ l.dep

$1x_1 + 4x_2 + 5x_3$

coef : 1 4 5

$$1\begin{bmatrix}1\\2\\3\end{bmatrix} + 1\begin{bmatrix}4\\5\\6\end{bmatrix} + (-1)\begin{bmatrix}5\\7\\9\end{bmatrix}$$

coef of vectors: 1, 1, -1

i.e. what coefficient means depends on the problem.

more column than rows

3 columns > 2 rows

$\Rightarrow$ free variable

$\left\{ \begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$ is linearly dependent since

$A\vec{x} = \vec{0}$

$$\begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has an infinite number of solutions

$$\left(\text{In particular } \begin{bmatrix} 9 & 4 & 3 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{3}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

or equivalently,

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} - \left(\frac{3}{2}\right) \begin{bmatrix} 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2) \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

or alternatively,

3 vectors in R^2 cannot be linearly independent.

$3 > 2 \Rightarrow$ lin dep

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \\ 7 \end{bmatrix} \right\}$ linearly independent?

$6 > 4$ at least 2 f.v.

NO

f.v.

Is $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$ linearly independent? NO

$\left\{ \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$ is lin ind

↓ E.F. to show you will see 3 pivots

Is $\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$ linearly independent? YES

$\begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} c_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = 0$

Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$ linearly independent? NO

f.v.

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & \boxed{5} \\ 0 & -3 & \boxed{-3} \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

EF

$$\begin{array}{c} f\,v \\ \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix} \\ f.v \end{array} \longrightarrow \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -3 \end{bmatrix} \\ f\,v \end{array}$$

$$\begin{bmatrix} 4 & 0 & 4 \\ 4-4 & 0-0 & -3-4 \\ 4-4 & 1-0 & 0-4 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & -7 \\ 0 & 1 & -4 \end{bmatrix}$$

$$\Big\downarrow R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 7 \end{bmatrix}$$

no f.v

$\Downarrow$

l indep