12:30 9/10

Does the pt $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ lie in plane spanned by $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$?

Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in the span of $\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\}$?

Yes, since $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \underbrace{x_1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix}}_{\text{linear combination}}$ has a solution.

Check:

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7 & 8 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 9 & 4 & 3 \\ 0 & 8 - \frac{7}{9}(4) & -5 - \frac{7}{9}(3) \end{array}\right]$$

Thus solution exists.

Short-cut: $span\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\} = R^2$

2-dim plane in $\mathbb{R}^2 = \mathbb{R}^2$

Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in $span \underbrace{\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}\right\}}_{\text{1-d line}}$?

NOT in span

Alg $\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 2 & -8 & -5 \end{array}\right]$

$\downarrow$

$\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 0 & 0 & * \end{array}\right]$

$* \neq 0$

No soln

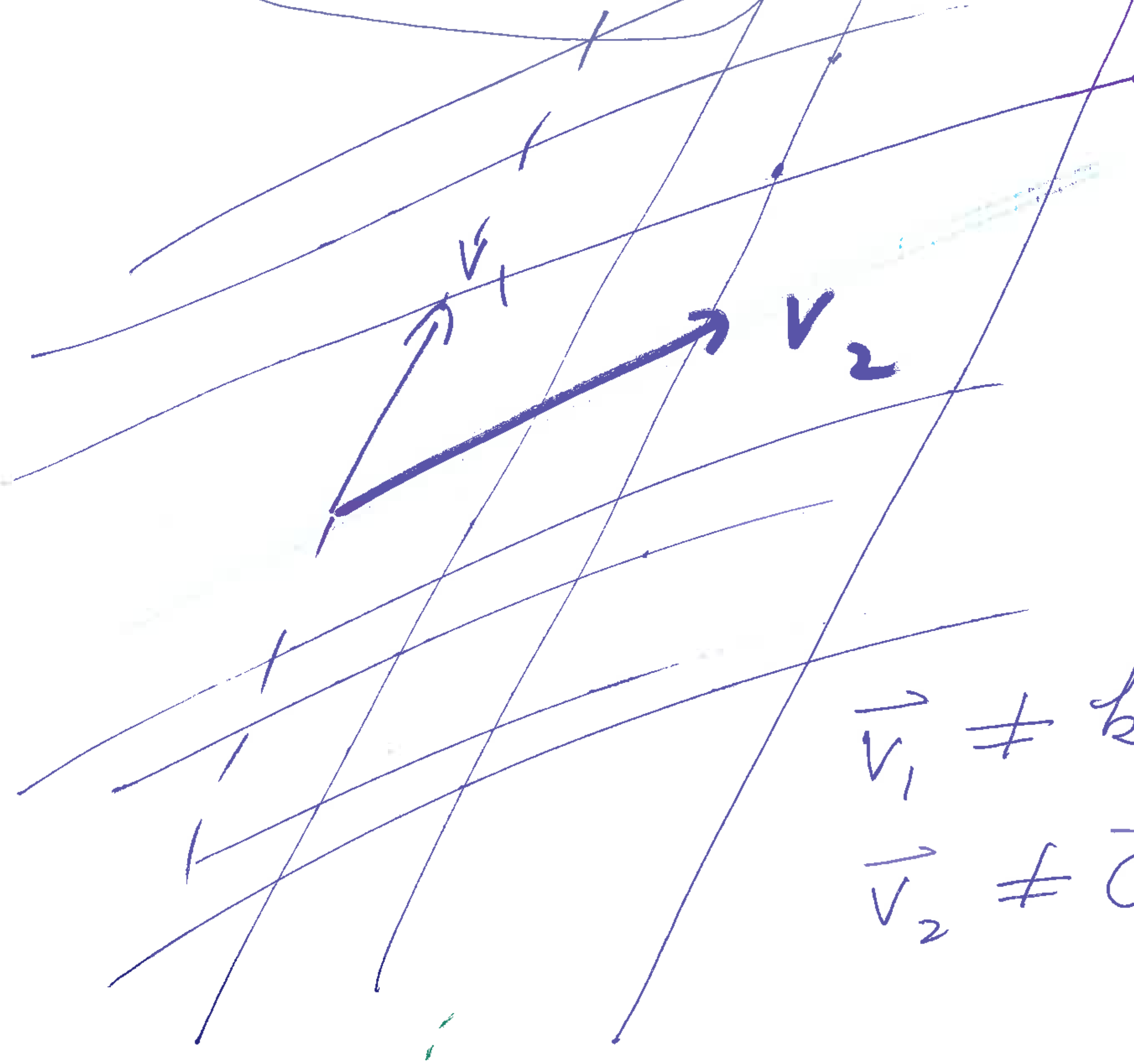
Not in span

Is $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ in $span \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}\right\}$?

YES

$\boxed{\begin{bmatrix} 10 \\ 20 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} -4 \\ -8 \end{bmatrix}}$

$$\mathrm{Span}\{\vec{v}_1, \vec{v}_2\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 \mid c_i \in \mathbb{R}\}$$



$$\vec{v}_1 \neq k\vec{v}_2$$

$$\vec{v}_2 \neq \vec{0}$$

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ in *span* $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix}\right\}$? YES

$$\left[\begin{array}{ccc|c}1 & 4 & 5 & 0\\2 & 5 & 7 & 3\\3 & 6 & 9 & 6\end{array}\right] \to \begin{bmatrix}1 & 4 & 5 & 0\\0 & -3 & -3 & 3\\0 & -6 & -6 & 6\end{bmatrix} \to \left[\begin{array}{ccc|c}1 & 4 & 5 & 0\\0 & -3 & -3 & 3\\0 & 0 & 0 & 0\end{array}\right]$$

E.F.

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ a lin comb of columns in coef matrix

∞ # Sol'n exists

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ in *span* $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$? YES

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ in *span* $\left\{\begin{bmatrix}10\\20\\30\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\}$? YES

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ in *span* $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}10\\20\\30\end{bmatrix}\right\}$? NO

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ in *span* $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix}\right\}$? NO

$\begin{bmatrix}1 & 4 & 5\\2 & 5 & 7\\3 & 6 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 4 & 5\\0 & -3 & -3\\0 & 0 & 0\end{bmatrix}$

BUT the columns span different spaces

Is $\begin{bmatrix}0\\3\\6\end{bmatrix}$ in *span* $\left\{\begin{bmatrix}0\\-1\\-2\end{bmatrix}\right\}$? YES

[Review 1.1 — 1.4]

Section 1.4

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

coef constants

create augmented matrix

↓

EF (to determine # of sol'ns)

↓

REF (to solve)

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

1.3 $A\vec{x} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

1.4, 1.5 $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\text{coef matrix } A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\text{constants}}$$

$A\vec{x} = \vec{a}_1 x_1 + \vec{a}_2 x_2 + \dots + \vec{a}_k x_k$

= linear combination of columns of A

Matrices as linear combinations:

1.1

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \cdot \\ \cdot \\ \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots, a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

system of equations

||

1.3

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix} x_2 + \ldots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

linear combination of vectors

||

1.4

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

Matrix equation

Solve:

REF

$$\underbrace{\begin{aligned} x_1 \quad + \qquad\qquad 6x_3 &= 7 \\ x_2 \quad + \quad 8x_3 &= 9 \end{aligned}}_{\text{coef}}$$

Solve:

REF

$$\overbrace{\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \end{bmatrix}}^{\text{coef}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

Augmented Matrix:

$$\begin{array}{c} x_1 + 6x_3 = 7 \\ x_2 + 8x_3 = 9 \end{array} \quad \begin{array}{c} \\ \end{array} \begin{array}{cccc} & & \text{f.v.} & \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \end{array}\right] \leftarrow \text{REF}$$

$$\boxed{\begin{aligned} x_1 &= -6x_3 + 7 \\ x_2 &= -8x_3 + 9 \\ x_3 &= x_3 \end{aligned}}$$

only free variables allowed of RHS

1.3

If possible write $\begin{bmatrix}7\\9\end{bmatrix}$ as a linear combination of

$\left[\begin{array}{ccc|c}1&0&6&7\\0&1&8&9\end{array}\right]$ REF

$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}$

columns of coef. matrix

$$\underline{7}\begin{bmatrix}1\\0\end{bmatrix}+\underline{9}\begin{bmatrix}0\\1\end{bmatrix}+\underline{0}\begin{bmatrix}6\\8\end{bmatrix}=\begin{bmatrix}7\\9\end{bmatrix}$$

E.F.

Is $\begin{bmatrix}7\\9\end{bmatrix}$ in $span\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\}$?

$\left[\begin{array}{ccc|c}1&0&6&7\\0&1&8&9\end{array}\right]$ (R)EF

No pivot

yes

choose a sol'n

Easiest soln let all free variable $=0$

Does $span\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\} = R^2$?

YES

$\left[\begin{array}{ccc|c}1&0&6&b_1\\0&1&8&b_2\end{array}\right]$ (R)EF

always have a sol'n

only need coeff

$\left[\begin{array}{ccc}1&0&6\\0&1&8\end{array}\right] \Big| \begin{array}{c}\ast\\\ast\end{array}$

pivot in every row of coef matrix $\Longrightarrow$ no pivot in last column of augment

1.3

If possible write $\begin{bmatrix} 7 \\ 9 \end{bmatrix}$ as a linear combination of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

Need a sol'n

REF

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \end{array}\right]$$

Let free variables $= 0$ (or whatever you want)

$x_1 = 7$, $x_2 = 9$, $x_3 = 0$ ← choose

$$7\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

EF

Is $\begin{bmatrix} 7 \\ 9 \end{bmatrix}$ in $span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \end{bmatrix}\}$?

YES

no pivot in last column of augmented

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \end{array}\right]$$

since sol'n exists

Does $span\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \end{bmatrix}\} = R^2$?

YES

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & b_1 \\ 0 & 1 & 8 & b_2 \end{array}\right]$$

augmented matrix

Actually only need

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 8 \end{bmatrix}$$

coef

since pivot in each row of coef matrix

Thus no pivot in last column of augmented

$0 \cdots 0 \mid *$

Not span all of R^2

1.5

$$\left[\begin{array}{cc|c} 1 & \boxed{2} & 0 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

Solve the following systems of equations:

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 = 0$

$0x_1 + 0x_2 = 0$

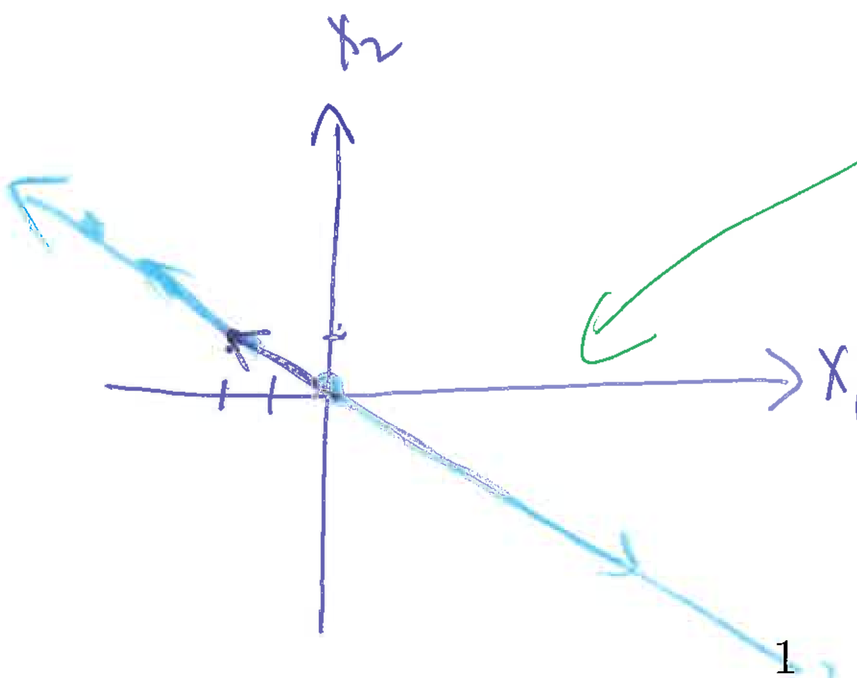
fv

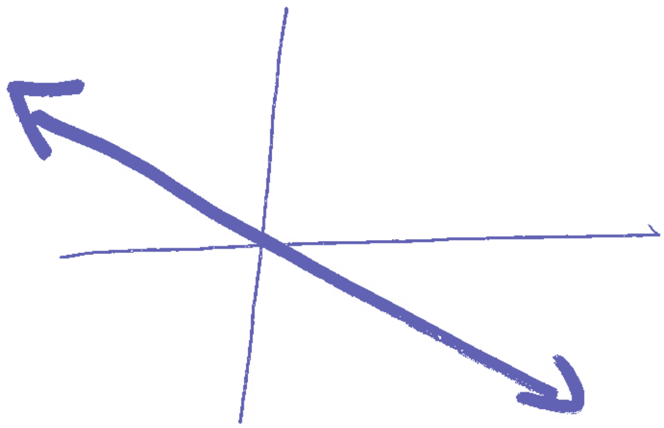
$x_1 = -2x_2$

$x_2 = x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2}$$

Solution set = line w/ slope $-1/2$ thru origin





$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} f$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} s$$

Sol'n is of form $\vec{v}\, t$ ← variable

↑ where $\vec{v}$ is a nonzero multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix} t$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix} x_2$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

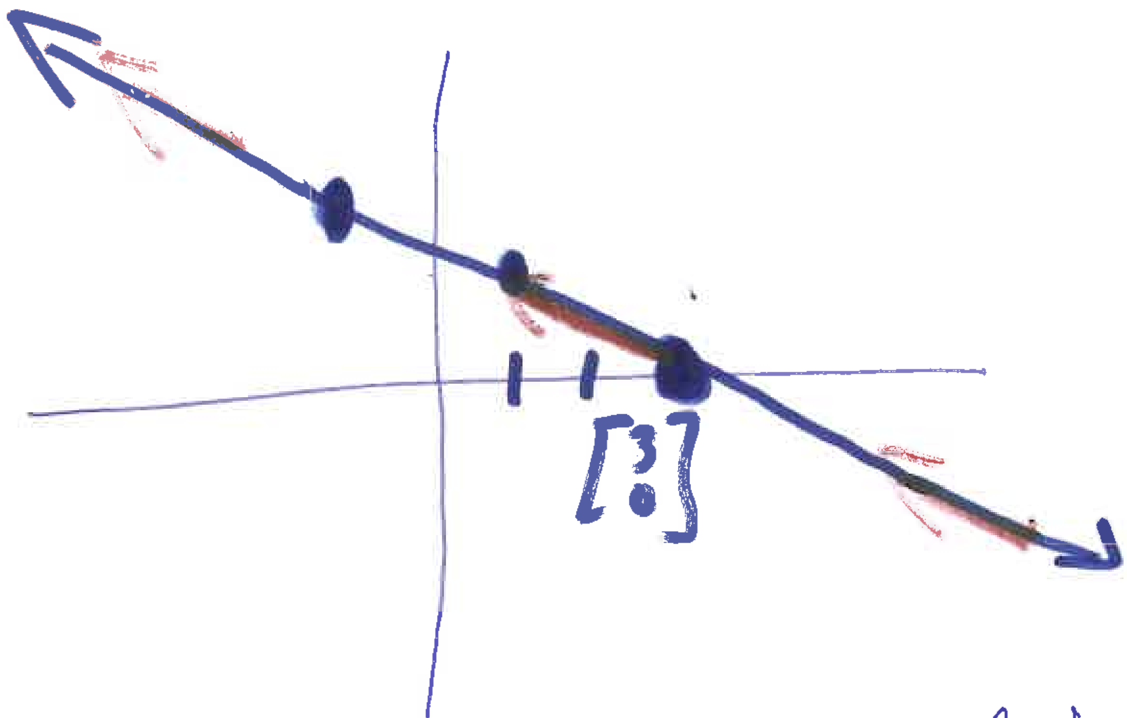
$$\left[\begin{array}{cc|c} 1 & \boxed{2} & 3 \\ 0 & \boxed{0} & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} -2x_2 + 3 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

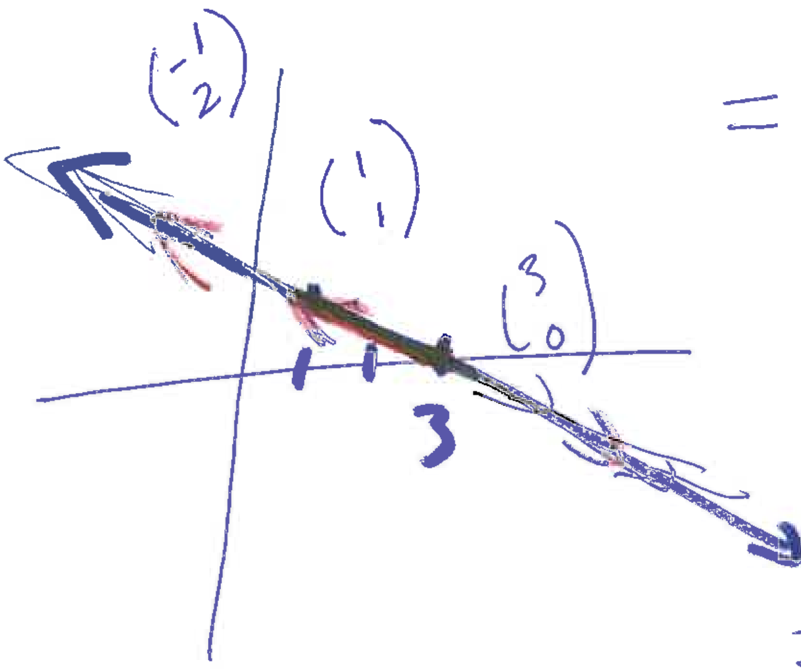
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Sol'n set = line
w/ slope $-1/2$
thru point $(3, 0)$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -6 \\ 3 \end{bmatrix}}_{\text{vector that lies on the line i.e. that is consistent w/ slope of line}} s + \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{\text{Any pt that lies on line}}$$

$$\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\boxed{\text{homogeneous}}$

Solution set:

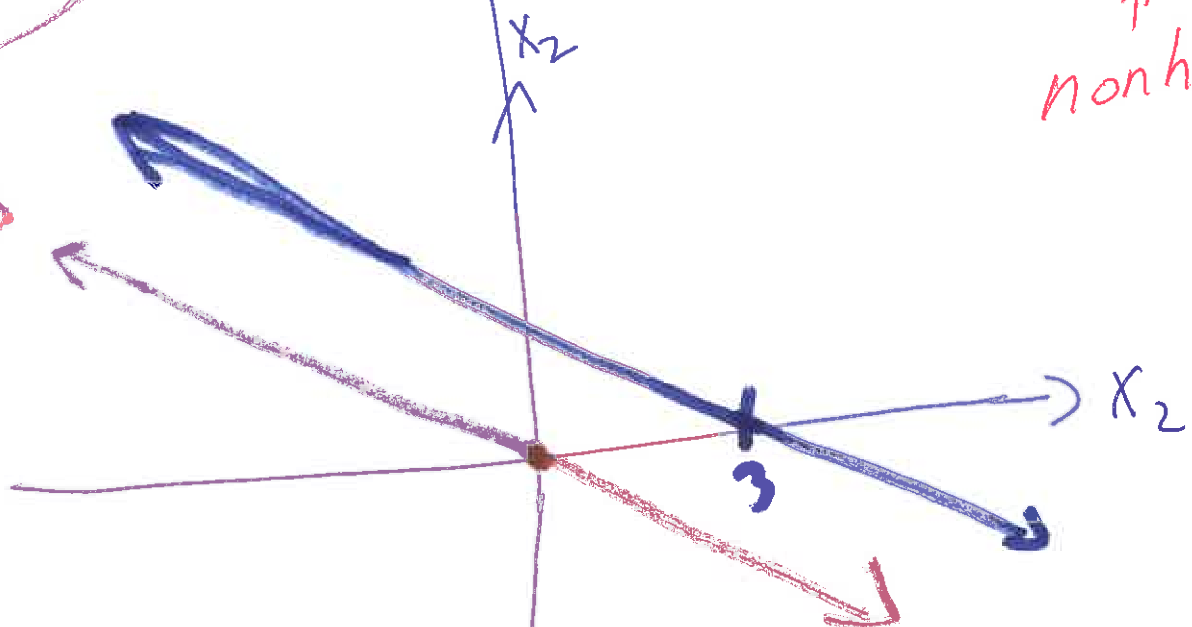
$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}x_2$$

$$\begin{bmatrix}1&2\\0&0\end{bmatrix}\vec{x}=\begin{bmatrix}3\\0\end{bmatrix}$$

$\boxed{\text{non homog}}$

sol'n set

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}x_2+\underbrace{\begin{bmatrix}3\\0\end{bmatrix}}_{\text{nonhom}}$$



<u>NOTE</u>: non homog sol'n = translated homog soln

= homog soln + $\begin{bmatrix}\text{A soln to non homog}\end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & \boxed{2} & \boxed{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -2x_2 - 3x_3$$

$$x_2 = x_2$$

$$x_3 = x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} x_3$$