1.3 Vectors in R^m

Defn: The vector **w** is a linear combination of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ if there exist scalars $c_1, \ldots, c_n$ such that

$$\boxed{\mathbf{w} = c_1\vec{\mathbf{v_1}} + c_2\vec{\mathbf{v_2}} + \ldots + c_n\vec{\mathbf{v_n}}}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$. [redacted]

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} c_1 + \begin{bmatrix} 4 \\ 8 \end{bmatrix} c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ is a linear comb of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ & $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$\Longleftrightarrow$ there exists soln for c_1 & c_2

$$\begin{bmatrix} 9c_1 \\ 7c_1 \end{bmatrix} + \begin{bmatrix} 4c_2 \\ 8c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 9c_1 + 4c_2 \\ 7c_1 + 8c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7 & 8 & -5 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7-\frac{7}{9}(9) & 8-\frac{7}{9}(4) & -5-\frac{7}{9}(3) \end{array}\right]$$

$$\downarrow \; R_2 - \frac{7}{9}R_1 \to R_2$$

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 0\left(\frac{9}{44}\right) & \frac{44}{9}\left(\frac{9}{44}\right) & -\frac{66}{9}\left(\frac{9}{44}\right) \end{array}\right]$$

EF sol'n exists

$$\downarrow \; R_2\left(\frac{9}{44}\right)$$

$$\left[\begin{array}{cc|c} 9 & 4-4 & 3-4\left(-3/2\right) \\ 0 & 1 & -3/2 \end{array}\right]$$

$$\downarrow \; R_1 - 4R_2$$

REF

$$\left[\begin{array}{cc|c} 9/9 & 0 & 9/9 \\ 0 & 1 & -3/2 \end{array}\right] \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= -3/2 \end{aligned}$$

Write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as lin comb

$$\begin{bmatrix} 9 \\ 7 \end{bmatrix} c_1 + \begin{bmatrix} 4 \\ 8 \end{bmatrix} c_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\boxed{\text{Ans: } \begin{bmatrix} 9 \\ 7 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}}$$

Check answer: $\begin{bmatrix} 9 \\ 7 \end{bmatrix} - \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$ ✓

augmented

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7 & 8 & -5 \end{array}\right] \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & \frac{44}{9} & -\frac{66}{9} \end{bmatrix} \to \begin{bmatrix} 9 & 4 & 3 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 9 & 0 & 9 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -\frac{3}{2} \end{array}\right]$$

summarizes

Found c_1 & c_2

Thus, $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} - (3/2) \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ ← wrote lin. comb.

If possible, write $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$$\underline{0}\begin{bmatrix} 9 \\ 7 \end{bmatrix} + \underline{0}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

$\times(-4)$

Not possible

If possible, write $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ as a linear comb'n of $\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -30 \\ 50 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 0\begin{bmatrix} -30 \\ 50 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -4 & 3 \\ 2^{-2} & -8^{+8} & -5^{-6} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 0 & -11 \end{array}\right]$$

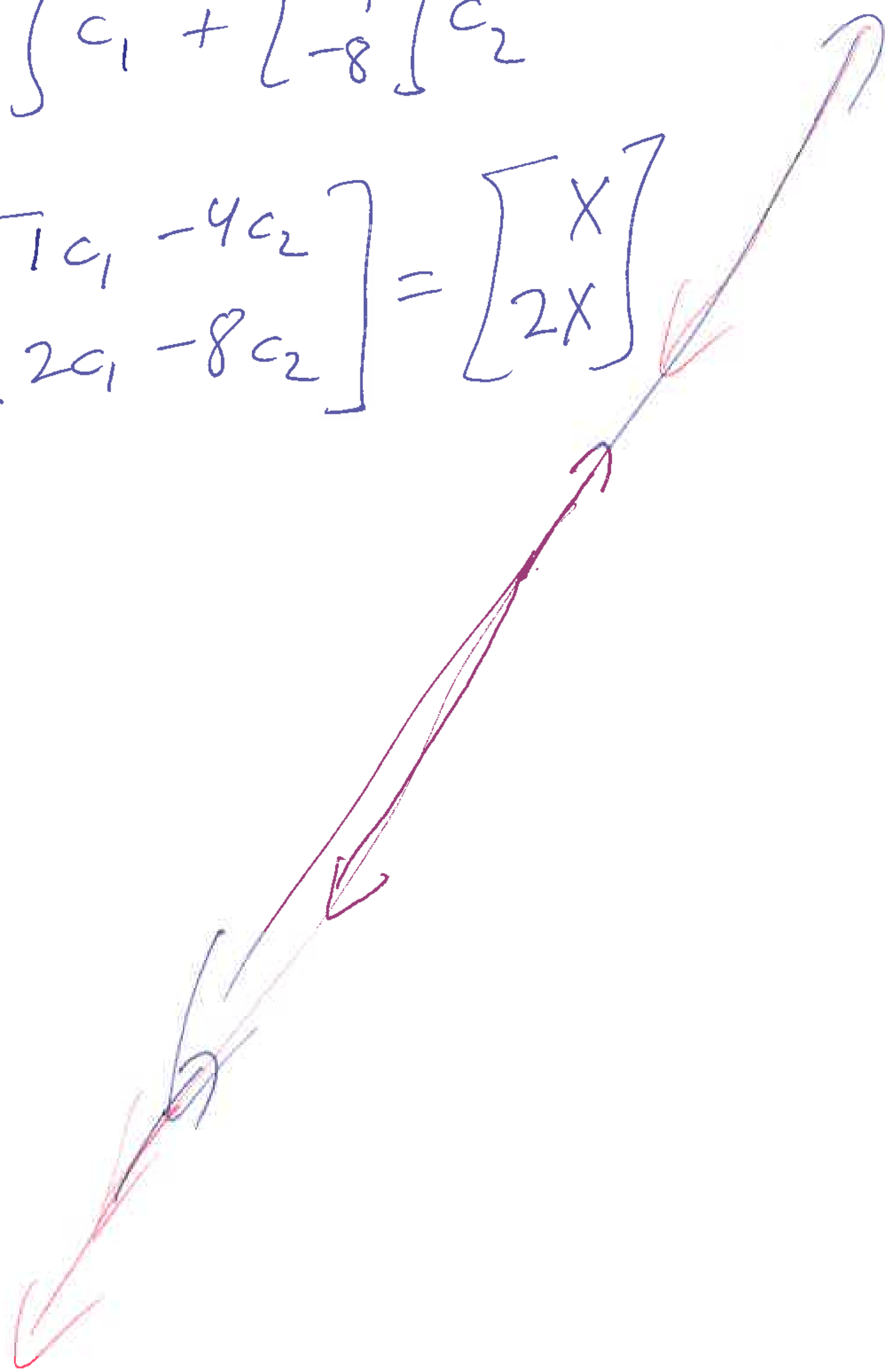
no sol'n

$\Updownarrow$

NOT POSSIBLE

$$\begin{bmatrix}1\\2\end{bmatrix}c_1 + \begin{bmatrix}-4\\-8\end{bmatrix}c_2$$

$$\begin{bmatrix}1c_1 - 4c_2\\ 2c_1 - 8c_2\end{bmatrix} = \begin{bmatrix}X\\2X\end{bmatrix}$$



Write $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} -30 \\ 50 \end{bmatrix}$

Long method

$$\left[\begin{array}{cc|c} 3 & -30 & 3 \\ -5 & 50 & -5 \end{array}\right]$$

$$\Big\downarrow \begin{matrix} R_1/3 \\ R_2/-5 \end{matrix}$$

$$\begin{bmatrix} 1 & -10 & 1 \\ 1 & -10 & 1 \end{bmatrix}$$

$$\Big\downarrow R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & -10 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$1c_1 - 10c_2 = 1$$

$$c_1 = -10c_2 + 1$$
$$c_2 = c_2$$

∞ # of sol'n

choose 1

$c_2 = 0 \Rightarrow c_1 = 1 = -10(0) + 1$

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 0 \begin{bmatrix} -30 \\ 50 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$c_2 = 1 \Rightarrow c_1 = -10(1) + 1 = -9$

$$-9 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + \begin{bmatrix} -30 \\ 50 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

Both answers are correct

$\boxed{\text{choose } \underline{\text{one}}}$

12:30/10:30

YES

If possible, write $\begin{bmatrix}0\\3\\6\end{bmatrix}$ as a l. c. of $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix}\right\}$ [redacted]

$$4\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} + 0\begin{bmatrix}5\\7\\9\end{bmatrix} = \begin{bmatrix}0\\3\\6\end{bmatrix}$$

same coef matrix

If possible, write $\begin{bmatrix}1\\0\\0\end{bmatrix}$ as a l. c. of $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}5\\7\\9\end{bmatrix}\right\}$ [redacted]

Not possible

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 4 & 5 & 0 & 1 \\ 2 & 5 & 7 & 3 & 0 \\ 3 & 6 & 9 & 6 & 0 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{ccc|cc} 1 & 4 & 5 & 0 & 1 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & -6 & -6 & 6 & -3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|cc} 1 & 4 & 5 & 0 & 1 \\ 0 & -3 & -3 & 3 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 4 & 5 & 0 & * \\ 0 & 1 & 1 & -1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

EF

free

$$\rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 4 & * \\ 0 & 1 & 1 & -1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

REF

$c_1 = -c_3 + 4$
$c_2 = -c_3 - 1$
$c_3 = c_3$

free

$c_1 = 4$
$c_2 = -1$
$c_3 = 0$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

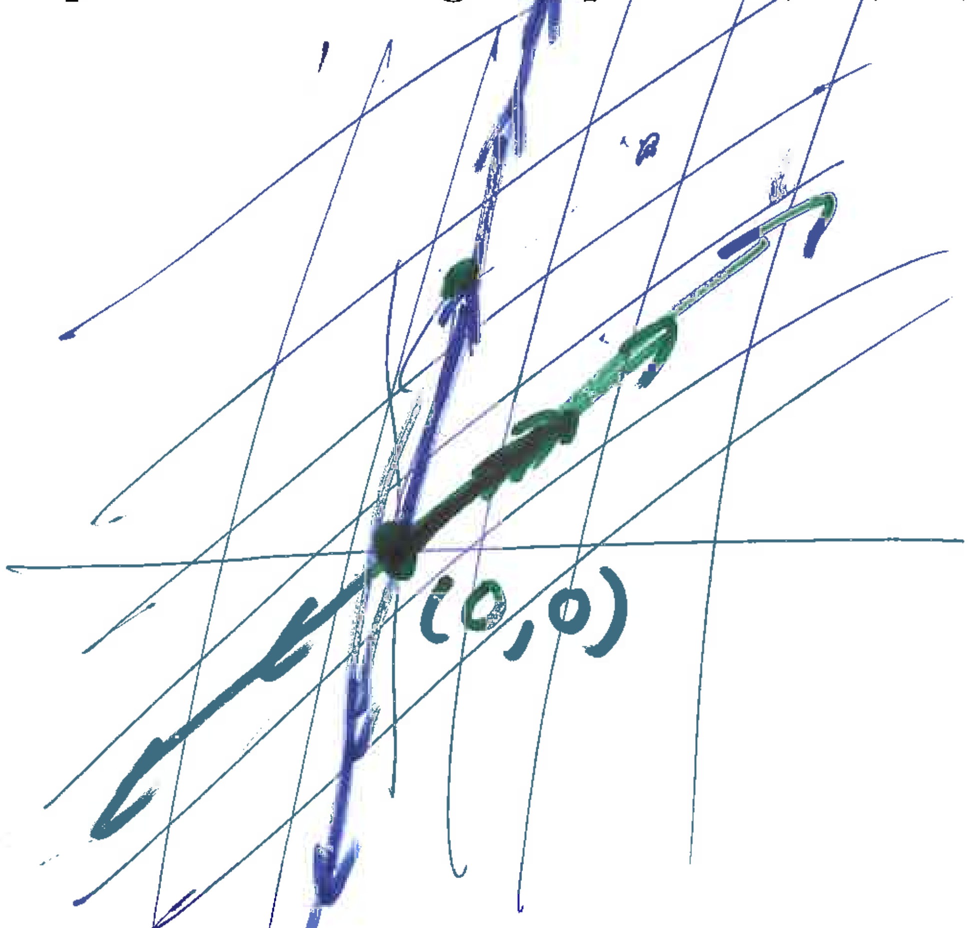
$$4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

Answer

$span\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$ = the set of all linear combinations, $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$, of the vectors in $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}\}$

= the hyperplane containing the vectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ anchored at $\mathbf{b} = \mathbf{0}$

= the hyperplane containing the points $\mathbb{0}, \mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$

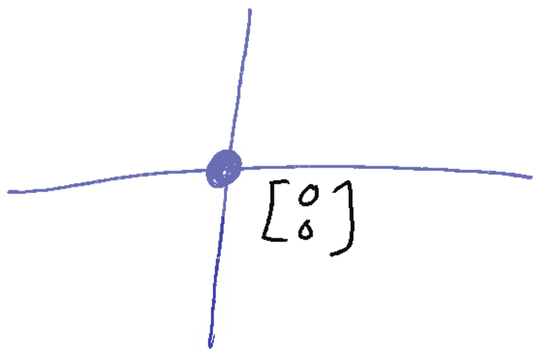


Let $A = [\mathbf{a_1} ... \mathbf{a_n}]$, where the a_i are k-vectors.

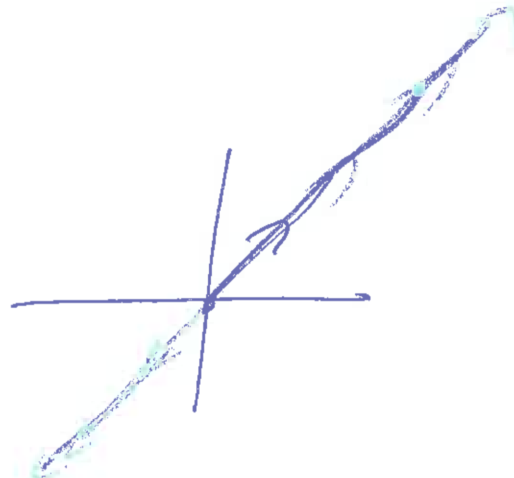
$\mathbf{b}$ is in $span\{\mathbf{a_1}, ..., \mathbf{a_n}\}$ if and only if $Ax = b$ has at least one solution.

$span\{\mathbf{a_1}, ..., \mathbf{a_n}\} = R^k$ if and only if $Ax = b$ has at least one solution for every $\mathbf{b}$
(leading entry in every row).

Span $\{v_1\}$



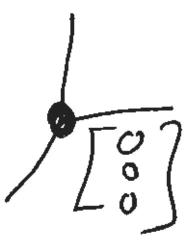
$\mathrm{span}\{\vec{0}\} = \{\vec{0}\}$



$\mathrm{span}\{\vec{v}\}$ $\quad \vec{v} \neq 0$

$= 1$-dim line

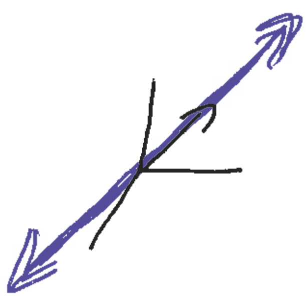
Span $\{\vec{v}_1, \vec{v}_2\}$



$\mathrm{span}\{\vec{0}, \vec{0}\}$

$= \vec{0}$

0-dim pt

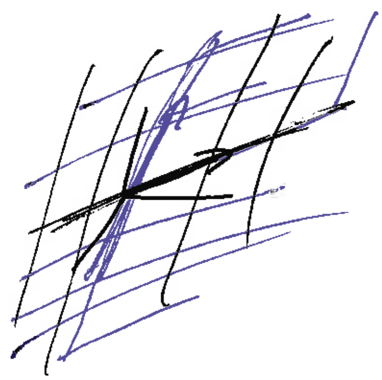


$\vec{v} \neq \vec{0}$

$\mathrm{span}\{\vec{v}, 2\vec{v}\}$

1-d

line



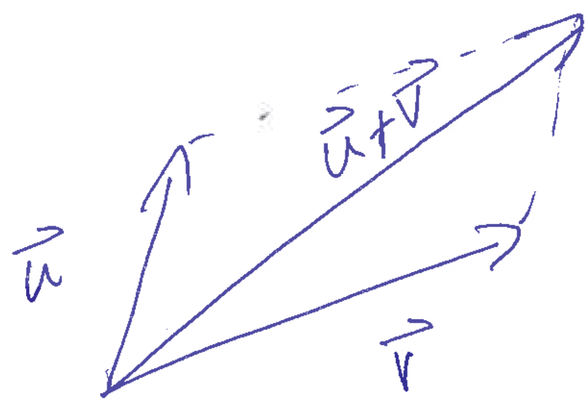
$\mathrm{span}\{v, w\}$

where $\vec{w}$ is not a multiple of $\vec{v}$ & $\vec{v} \neq \vec{0}$

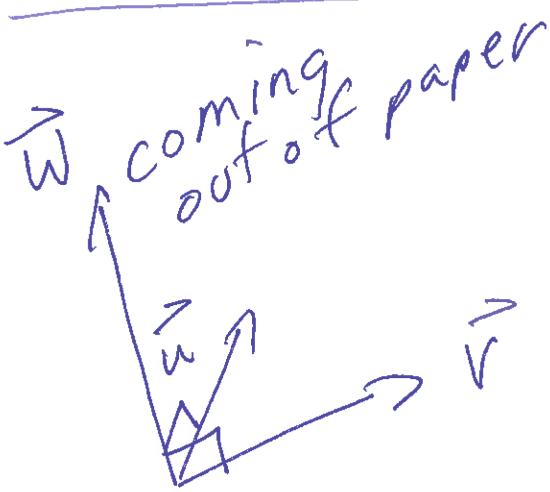
2-d

plane

Span $\{v_1, v_2, v_3\}$ will be at most 3-dim



span$\{\vec{u}, \vec{v}, \vec{u}+\vec{v}\}$ $\longrightarrow$ 2-dim plane



span $\{u, v, w\}$ 3-dim

$\text{span}\{v_1, v_2, v_3\}$

0-dim
point

1-dim
line
all mult
of each
other

ex $\text{span}\{v, w, v+w\}$
2-d
plane

$\text{span}\{v_1, v_2, v_3\}$
3-dim

$\text{Span}\{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^n$
will be atmost 4-dim
hyperplane

span $\{v_1, \ldots v_K\}$ will be at most K-dim

How does this relate to pivots

dim = # of pivots in coef matrix for span of the columns of coef matrix

Does $span\{\begin{bmatrix}9\\7\end{bmatrix}, \begin{bmatrix}4\\8\end{bmatrix}\} = R^2$? Yes, since

$$x_1\begin{bmatrix}9\\7\end{bmatrix} + x_2\begin{bmatrix}4\\8\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \text{ has a sol'n for all } \begin{bmatrix}b_1\\b_2\end{bmatrix}.$$

I.e., $\begin{bmatrix}9 & 4\\7 & 8\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ has a sol'n for all $\begin{bmatrix}b_1\\b_2\end{bmatrix}$.

Check:

pivot   pivot

$$\left[\begin{array}{cc|c}9 & 4 & b_1\\7 & 8 & b_2\end{array}\right] \rightarrow \left[\begin{array}{cc|c}9 & 4 & b_1\\0 & 8-\frac{7}{9}(4) & \underbrace{b_2-\frac{7}{9}(b_1)}_{\text{no pivots}}\end{array}\right] \Rightarrow \text{sol'n exists}$$

Thus solution exists no matter what b_1 and b_2 are.

Short-cut: $\begin{bmatrix}4\\8\end{bmatrix}$ is not a multiple of $\begin{bmatrix}9\\7\end{bmatrix}$.

Thus span of $\{\begin{bmatrix}9\\7\end{bmatrix}, \begin{bmatrix}4\\8\end{bmatrix}\}$ is 2-dimensional.

The only 2-dimensional plane in R^2 is R^2.

Note this short-cut only works in R^2

Also works in R^n to determine if span is 2-dim if given 2 vectors

$$\text{Does } span \underbrace{\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\}}_{\text{1-dim line}} = R^2?$$

$\times(-4)$

NO

$$\text{Does } span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \\ 9 \end{bmatrix} \right\} = R^4?$$

NO

2-dim

3 vectors can't span 4-d space

$$\text{Does } span\left\{ \cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \cancel{\begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 10 \\ 4 \\ -3 \end{bmatrix} \right\}$$ [redacted]

$\neq R^3?$ NO

$$\left(span \left\{ \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\} \right)$$

$$\begin{array}{c|c} \begin{bmatrix} \cancel{0} & 2 & \cancel{4} & 0 & 6 & 10 \\ \cancel{0} & 2 & \cancel{4} & -1 & 2 & 4 \\ \cancel{0} & -3 & \cancel{-6} & 2 & -1 & -3 \end{bmatrix} & \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \end{array}$$

coef

is row equivalent to

$$\begin{array}{c|c} \begin{bmatrix} 0 & \boxed{1} & 2 & \boxed{0} & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} b_1 \\ \sim \\ \sim \end{matrix} \end{array}$$

EF

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 4 & 0 & 6 & 10 \\ 0 & 2 & 4 & -1 & 2 & 4 \\ 0 & -3 & -6 & 2 & -1 & -3 \end{bmatrix}$$

$$3\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + 4\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$$

$$5\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + 6\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ -3 \end{bmatrix}$$

lie in plane spanned by

$$\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$\mathbb{R}^2$

Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in the span of $\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\}$? YES

Yes, since $\begin{bmatrix} 3 \\ -5 \end{bmatrix} = x_1 \begin{bmatrix} 9 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ has a solution.

Check:

$$\left[\begin{array}{cc|c} 9 & 4 & 3 \\ 7 & 8 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 9 & 4 & 3 \\ 0 & 8 - \frac{7}{9}(4) & -5 - \frac{7}{9}(3) \end{array}\right]$$

Thus solution exists.

Short-cut: $span\{\begin{bmatrix} 9 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}\} = R^2$

Is $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ in $span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\}$? NO

Is $\begin{bmatrix} 10 \\ 20 \end{bmatrix}$ in $span \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix} \right\}$? YES

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$? YES

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 2 & 5 & 7 & 3 \\ 3 & 6 & 9 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -6 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

sol'n exists

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$? YES

=

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$? YES

1-dim

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \right\}$? NO

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}$? NO

Is $\begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$?