To solve a system of equations:

i.) Create augmented matrix.

ii.) Put matrix into EF. if needed ⇐ can now determine # of sol'ns

iii.) Put into REF. if needed

iv.) Solve.

Stop once you know answer

Case 1: If pivot in last column of augmented matrix.

$\Longleftrightarrow$ ~~Then~~ system of equations has **no solution**.

EX: $0x + 0y + 0z = 6$

$0 = 6 \Rightarrow$ no sol'n

Case 2: If no pivot in last column of augmented matrix:

a.) No free variables implies **unique solution**.

b.) Free variables imply an **infinite number of solutions**
Solve for pivot column variables in terms of free variables.

Solve:

$$3x + 6y + 9z = 0$$
$$4x + 5y + 6z = 3$$
$$7x + 8y + 9z = 0$$

$$\left[\begin{array}{ccc|c} 3/3 & 6/3 & 9/3 & 0/3 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

$R1 \to R1/3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4-4 & 5-8 & 6-12 & 3-0 \\ 7-7 & 8-14 & 9-21 & 0-0 \end{array}\right]$$

$R_2 - 4R_1$

$R_3 - 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 3 \\ 0 & -6 & -12 & 0 \end{array}\right]$$

$R_3 - 2R_2$

$R_2/-3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

EF

$0x + 0y + 0z = 6$

$0 = 6$

No soln

$R_2 - R_1 \to R_1$

affects determinant

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 4 & 5 & 6 & 3 \\ 7 & 8 & 9 & 0 \end{array}\right]$$

optional method

1.5: A system of equations is **homogeneous** if $b_i = 0$ for all i.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right]$$

A homogeneous system of LINEAR equations can have

a.) Exactly one solution ($\mathbf{x} = \mathbf{0}$)

b.) Infinite number of solutions (including, of course, $\mathbf{x} = \mathbf{0}$).

Solve:

$$\begin{aligned} 3x + 6y + 9z &= b_1 \\ 4x + 5y + 6z &= b_2 \\ 7x + 8y + 9z &= b_3 \end{aligned}$$

homog

where 1a.) $b_1 = 0,\ b_2 = 0,\ b_3 = 0$

non homog

1b.) $b_1 = 0,\ b_2 = 3,\ b_3 = 0$

1c.) $b_1 = 6,\ b_2 = 5,\ b_3 = 8$

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 9 & 0 & 0 & 6 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

EF work down and left to right

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 9 & 0 & 0 & 6 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

$$\downarrow \tfrac{1}{3}R_1 \rightarrow R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 4 & 5 & 6 & 0 & 3 & 5 \\ 7 & 8 & 9 & 0 & 0 & 8 \end{array}\right]$$

$$\downarrow R_2 - 4R_1 \rightarrow R_2, \quad R_3 - 7R_1 \rightarrow R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & -6 & -12 & 0 & 0 & -6 \end{array}\right]$$

$$\downarrow R_3 - 2R_2 \rightarrow R_3$$

EF

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 0 & 2 \\ 0 & -3 & -6 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

see column 5

0 = -6

b) no sol'n

↓ already know sol'n to system b.

a c

$$\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

a, c have ∞ # of sol'ns

free variable col'mn

$$\downarrow -\tfrac{1}{3}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - 2R_2 \rightarrow R_1} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

EF → REF

work up & right to left

REF

$$\begin{array}{c}
\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & 2 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\
\begin{array}{ccc} x & y & z \end{array}
\end{array}$$

coefficients = constants colums

pivot column

free variable column

Solve for pivot column variable in terms of free variables

Solve for x & y in terms of z

b) no sol'n

$$\left[\begin{array}{ccc|cc} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) (c)

$x - z = 0$
$y + 2z = 1$

$x - z = 0$
$y + 2z = 0$

a) $x = z$
$y = -2z$
$z = z$

section 1.5

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} z$$

c) $x = z + 0$
$y = -2z + 1$
$z = z + 0$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -2z+1 \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} z + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{nonh}}$$

REF

$$\begin{bmatrix} 1 & 0 & 5 & 7 & 0 & 0 & 2 \\ 0 & 1 & 3 & 4 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} x & y & z & w & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

REF

$x = 2$
$y = 8$
$z = 4$
$w = 0$

prob A

REF

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \end{array}$$
$$\left[\begin{array}{ccccccc|c} 1 & 4 & 0 & 0 & 3 & 7 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 & 2 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

fv

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

REF

$x = -3z + 2$
$y = -5z + 8$
$z = z$

$$\begin{bmatrix} 0 & 0 & 1 & 7 & 2 \\ 0 & 1 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

not EF

$$\begin{bmatrix} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -4x_2 - 3x_5 - 7x_6 + 2$$
$$x_2 = x_2$$
$$x_3 = -5x_5 - 2x_6 + 8$$
$$x_4 = -3x_5 - x_6 + 4$$
$$x_5 = x_5$$
$$x_6 = x_6$$
$$x_7 = 0$$

prob A answer

Determine if the augmented matrix is in echelon form. If it is, determine if the corresponding system of equations has no solution, exactly one solution, or an infinite number of solutions. If it has an infinite number of solutions, state the dimension of the hyperplane of the solutions.

EF
$$\left[\begin{array}{cccc|c} 5 & 6 & 3 & 7 & 2 \\ 0 & 7 & 5 & 2 & 8 \\ 0 & 0 & 4 & 1 & 4 \end{array}\right]$$
∞

$$\left[\begin{array}{ccccc} 0 & 6 & 3 & 7 & 2 \\ 0 & 7 & 5 & 2 & 8 \\ 0 & 0 & 4 & 1 & 4 \end{array}\right]$$
not ef

EF
$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$
0's

No sol'n

EF
$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$
∞

EF
$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
∞

$$\left[\begin{array}{cccc|c} 0 & 6 & 3 & 7 & 2 \\ 0 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
EF

∞

Rows of all zero's are at the bottom

$x_1 = x_1$

$x_2 =$

$x_3 =$

$x_4 = 4$

EF

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \end{array}\right]$$

no free variable

unique sol'n

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

EF

$5x_4 = 0$

$\Rightarrow x_4 = 0$

EF

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

no sol'n

$0 = 3$

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 7 & 2 \\ 0 & 9 & 5 & 2 & 8 \\ 0 & 0 & 7 & 1 & 4 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

EF

$$\begin{bmatrix} 0 & 6 & 3 & 7 & 2 \\ 7 & 0 & 5 & 2 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 8 \end{bmatrix}$$

$$\underbrace{x + 2y + 3z = 4}_{\text{plane}}$$

1.3 Vectors in R^m

Defn: $\mathbf{u} = (u_1, ..., u_m)$, $\mathbf{v} = (v_1, ..., v_m)$ are **vectors** in $\mathbf{R^m}$.

Defn: $u_1, ..., u_m$ are the **components** of $\mathbf{u}$.

Defn: $\mathbf{u} = \mathbf{v}$ if and only if $u_i = v_i$ for all i.

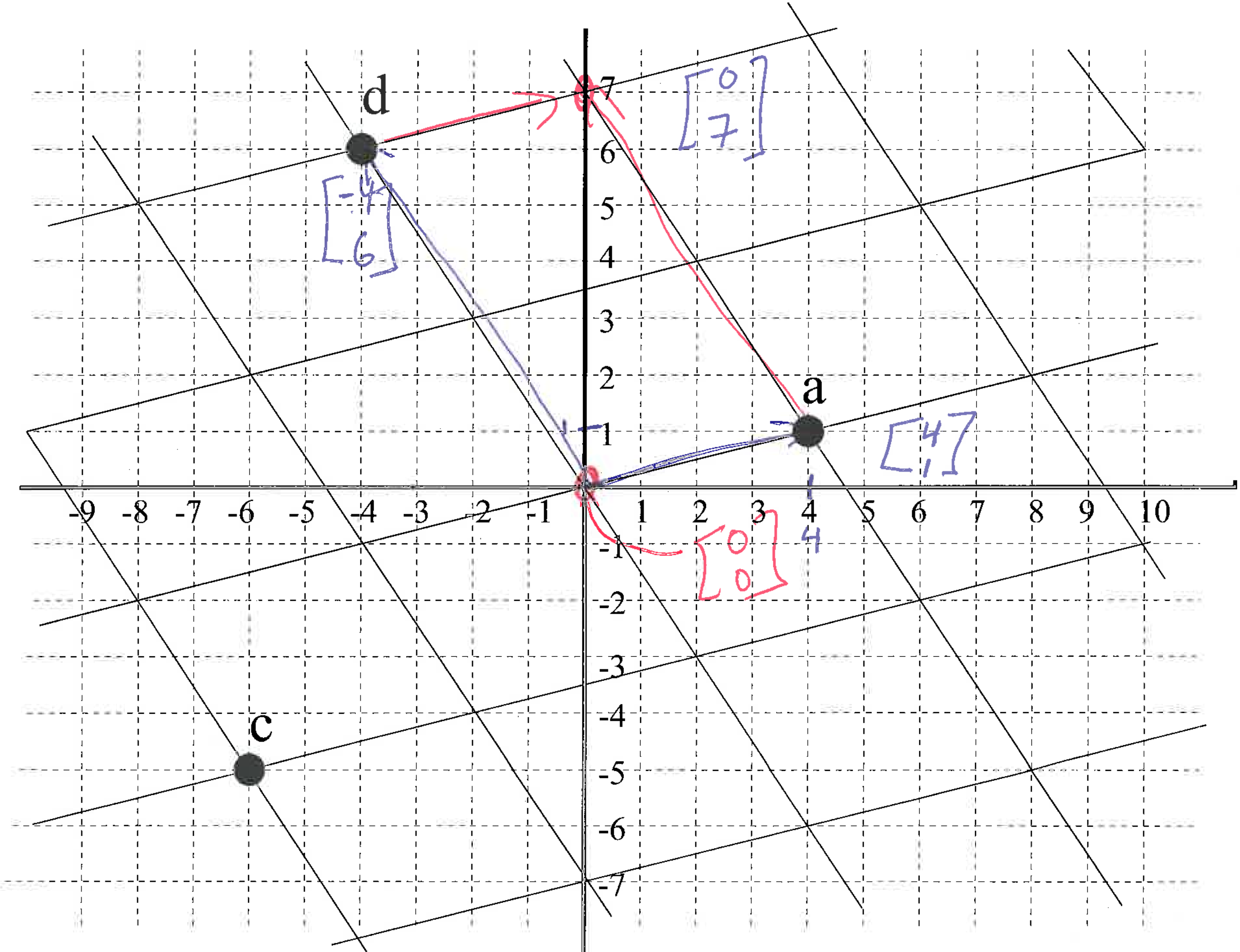
Defn: The **zero vector** in $\mathbf{R^m}$ is the m-vector $\mathbf{0} = (0, 0, ..., 0)$.

Vector Addition

Defn: The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = (u_1 + v_1, ..., u_m + v_m)$.

Defn: The **negative** of $\mathbf{u}$ is the vector $-\mathbf{u} = (-u_1, ..., -u_m)$

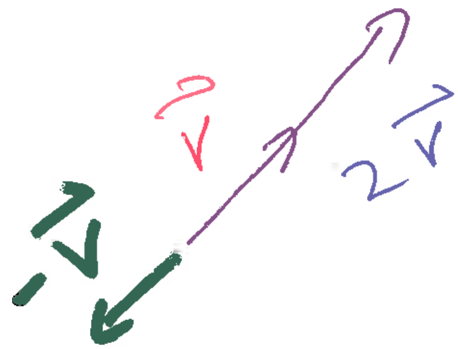
Defn: The **difference** between $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, ..., u_m - v_m)$.



d
a
c

Defn: In this class a **scalar**, c, is a real number.

Defn: The **scalar multiple** of $\mathbf{u}$ by c is the vector $c\mathbf{u} = (cu_1, ..., cu_m)$.



Thm: The vectors, $\mathbf{u}$ and $\mathbf{v}$, are collinear iff there exists a scalar c such that $\mathbf{v} = c\mathbf{u}$. In this case

a.) if $c > 0$, $\mathbf{u}$ and $c\mathbf{u}$ have the same direction.

b.) If $c < 0$, $\mathbf{u}$ and $c\mathbf{u}$ have opposite directions.

Defn: The *length (norm, magnitude)* of $\mathbf{u}$ is its distance from $\mathbf{0}$ and is denoted by

$$||\mathbf{u}|| = d(\mathbf{0}, \mathbf{u}) = \sqrt{u_1^2 + u_2^2 + ... + u_m^2}.$$

Two vectors are equivalent if they have the same direction and length.