6.1: Inner Products.

Defn: Let V be a vector space over the real numbers. An **inner product** for V is a function that associates a real number $\mathbf{u} \cdot \mathbf{v}$ to every pair of vectors, $\mathbf{u}$ and $\mathbf{v}$ in V such that the following properties are satisfied for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalars c:

a.) $\mathbf{u} \cdot \mathbf{v} \ = \ \mathbf{v} \cdot \mathbf{u}$

b.) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) \ = \ \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

c.) $(c\mathbf{u})\cdot v \ = \ c(\mathbf{u} \cdot \mathbf{v}) \ = \ \mathbf{u}\cdot(c\mathbf{v})$

d.) $\mathbf{u} \cdot \mathbf{u} \ \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

A vector space V together with an inner product is called an **inner product space.**

Thm 6.1.1$'$: Let V be an inner product space. Then for all vectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{v}$ in V and scalars c_1, c_2:

a.) $(c_1\mathbf{u_1} + c_2\mathbf{u_2})\cdot\mathbf{v} \ = \ \mathbf{v} \cdot (c_1\mathbf{u_1} + c_2\mathbf{u_2})$
$$= c_1(\mathbf{u_1} \cdot \mathbf{v}) + c_2(\mathbf{u_2} \cdot \mathbf{v})$$

b.) $\mathbf{0} \cdot \mathbf{v} \ = \ \mathbf{v} \cdot \mathbf{0} \ = \ 0$

Inner Product Example: Dot product on R^n.

Defn: $\Sigma_{k=1}^m a_k = a_1 + a_2 + ... + a_m$

Defn:
The **dot product** of $\mathbf{u} = (u_1, ..., u_m)$ & $\mathbf{v} = (v_1, ..., v_m)$ is

$$\mathbf{u} \cdot \mathbf{v} = \Sigma_{k=1}^m u_k v_k.$$

In words, $\mathbf{u} \cdot \mathbf{v}$ is the sum of the products of the corresponding components of $\mathbf{u}$ and $\mathbf{v}$.

Note that $\mathbf{u} \cdot \mathbf{v}$ is a real number (not a vector).

Examples:

$(1, 2, 3) \cdot (4, 5, 6) =$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} =$

its

Defn: Let $\mathbf{v}$ be a vector in an inner product space $\mathbf{V}$. The **length** or **norm** of $\mathbf{v} = ||\mathbf{v}|| = \sqrt{< \mathbf{v}, \mathbf{v} >}$.

$||(3, 4)|| =$

Defn: The vector $\mathbf{u}$ is a **unit vector** if $||\mathbf{u}|| = 1$.

Note that $\frac{\mathbf{v}}{||\mathbf{v}||}$ is a unit vector.

Create a unit vector in the direction of the vector (3, 4):

Create a unit vector in the direction of the vector (1, 2):

Create a unit vector in the direction of the vector (-2, 1):

Defn: **u** and **v** are **orthogonal** (or **perpendicular**) if

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Example: $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1(-2) + 2(1) = 0$

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

Example: $\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \\ \frac{2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \\ \frac{1}{\sqrt{5}} \end{bmatrix} =$

Thus $\left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} \frac{-2}{\sqrt{5}} \\ \\ \frac{1}{\sqrt{5}} \end{bmatrix} \right\}$ is a set of orthogonal unit vectors.

Observation: $\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} =$

Suppose $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ is a pair of orthogonal unit vectors. Then

$$\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{u}\cdot\mathbf{v} \\ \mathbf{v}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix} =$$