5.3: Diagonalization

Note that multiplying diagonal matrices is easy:

Let $D = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$. Then

$$D^2 = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} =$$

$$D^k =$$

Defn: The matrices A and B are **similar** if there exists an invertible matrix P such that $B = P^{-1}AP$.

Defn: A matrix A is **diagonalizable** if A is similar to a diagonal matrix.

I.e. A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

Application: Calculating A^k.

$$P^{-1}AP \quad = \quad D$$

$k = 1: \ A =$

$k = 2: \ A^2 = PDP^{-1}PDP^{-1}$

$k = 3: \ A^3 = PDP^{-1}PDP^{-1}PDP^{-1}$

Similarly $A^k =$

Example:

Let $D = \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix}$

Then $\begin{bmatrix} -1 & 0 \\ -55 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$

Thus, $A^3 =$

Equivalent Questions:

- Given an $n \times n$ matrix, does there exist a basis for R^n consisting of eigenvectors of A?
- Given an $n \times n$ matrix, does there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?

Thm: Let A be an $n \times n$ matrix. The following are equivalent:

a.) A is diagonalizable.

b.) A has n linearly independent eigenvectors.

c.) There exists a basis for R^n consisting of eigenvectors of A.

Example: Suppose $AP = PD$ where

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix}$$

Then $PD = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 6 \end{bmatrix} =$

Thus $AP = A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =$

Hence $A \begin{bmatrix} 1 \\ 3 \end{bmatrix} =$ and $A \begin{bmatrix} 2 \\ 4 \end{bmatrix} =$

Thus an eigenvalue of A = ___ with eigenvector

Another eigenvalue of A = ___ with eigenvector

Thus if $AP = PD$, then
if the diagonal entries of D are $d_1, ..., d_n$

and the i^{th} column of P is an ______________________

corresponding to the eigenvalue ______.

Note P is an invertible SQUARE matrix where columns P are _________________ of the matrix A

To diagonalize a matrix A:

1.) Find the eigenvalues of A.

Solve $det(\lambda I - A) = 0$ for λ.

2.) Find a basis for each of the eigenspaces.

Solve $(\lambda_j I - A)\mathbf{x} = 0$ for $\mathbf{x}$.

Case 3a.) IF the geometric multiplicity is LESS then the algebraic multiplicity for at least ONE eigenvalue of A, then A is NOT diagonalizable. (Cannot find square matrix P).

Case 3b.) IF the geometric multiplicity equals the algebraic multiplicity for ALL the eigenvalues of A, then A is diagonalizable. Thus,

• Use the eigenvalues of A to construct the diagonal matrix D

• Use the basis of the corresponding eigenspaces for the corresponding columns of P. (NOTE: P is a SQUARE matrix).

NOTE: ORDER MATTERS.

Examples:

$$A = \begin{bmatrix} 4 & 1 \\ 7 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Thm: Suppose λ_i, $i = 1, ..., n$ are DISTINCT eigenvalues of a matrix A. If $\mathcal{B}_i$ is a basis for the eigenspace corresponding to λ_i, then

$\mathcal{B} = \mathcal{B}_1 \cup ... \cup \mathcal{B}_n$ is linearly independent.

Defn: Suppose the characteristic polynomial of A is

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2}...(\lambda - \lambda_n)^{k_n}$$

where the λ_i, $i = 1, ..., n$ are DISTINCT. Then the **algebraic multiplicity of** λ_i is k_i.

That is the **algebraic multiplicity of** λ_i is the number of times that $(\lambda - \lambda_i)$ appears as a factor of the characteristic polynomial of A.

Defn: The **geometric multiplicity of** λ_i = dimension of the eigenspace corresponding to λ_i.

Thm (Geometric and Algebraic Multiplicity):

a.) The geometric multiplicity is less than or equal to the algebraic multiplicity [That is, Nullity of $(\lambda_i I - A) \leq k_i$].

b.) A is diagonalizable if and only if the geometric multiplicity is equal to the algebraic multiplicity for every eigenvalue.