2.8 Subspaces of R^n.

Example: The **nullspace of** A is the solution set of $A\mathbf{x} = \mathbf{0}$.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4]{}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - R_2 \to R_3]{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Nullspace of } A = \text{Solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{0}$$

$$= \text{solution space of } \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$= \text{solution space of } \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Suppose $A\mathbf{v_1} = \mathbf{0}$ and $A\mathbf{v_2} = \mathbf{0}$, then $A(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = \mathbf{0}$

NOTE: Nullspace of A = span{ }

2.8 Subspaces of R^n.

Long definition emphasizing important points:

Defn: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if and only if the following three conditions are satisfied:

i.) $\mathbf{0}$ is in W,

ii.) if $\mathbf{v_1}, \mathbf{v_2}$ in W, then $\mathbf{v_1} + \mathbf{v_2}$ in W,

iii.) if $\mathbf{v}$ in W, then $c\mathbf{v}$ in W for any scalar c.

Short definition: Let W be a nonempty subset of R^n. Then W is a subspace of R^n if $\mathbf{v_1}, \mathbf{v_2}$ in W implies $c_1\mathbf{v_1} + c_2\mathbf{v_2}$ in W,

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Note that if S is a subspace, then

if $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ in S, then $c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_n\mathbf{v_n}$ is in S.

$0\mathbf{v} = \mathbf{0}$ is in S.

Defn: Let S be a subspace of R^k. A set $\mathcal{T}$ is a **basis** for S if
i.) $\mathcal{T}$ is linearly independent and
ii.) $\mathcal{T}$ spans S.

Examples: Nullspace and Column Space.

Let $A = [\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}]$, a $k \times n$ matrix.

Defn: The **column space of** $A = span\{\mathbf{c_1}, \mathbf{c_2}, ..., \mathbf{c_n}\}$

Thm: The column space of A is a subspace of R^k

Note: Suppose B is row equivalent to A, then the column space of B need not be the same as the column space of A.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow[R_2 - 2R_1 \to R_2,\ R_3 - 3R_1 \to R_3,\ R_4 - R_1 \to R_4]{}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 - R_2 \to R_3]{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{The column space of } A = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix} \right\}$$

$= \text{span} \{ \qquad\qquad\qquad\qquad \}.$

Thus a basis for the column space of A is $\{ \qquad\qquad\qquad\qquad \}.$ ∎

Note we took the leading entry columns in the ORIGINAL matrix.

Why are we so interested in the column space?

Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 6 \\ 9 \\ 3 \end{bmatrix} x_3 + \begin{bmatrix} 4 \\ 2 \\ 12 \\ 4 \end{bmatrix} x_4 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a sol'n?

Does $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ have a solution?

Is $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \\ 2 \end{bmatrix} \right\}$ = column space of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix}$?

Example 1: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 22 \\ 31 \\ 9 \end{bmatrix}$ have a sol'n?

Example 2: Does $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 3 & 7 & 9 & 12 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 8 \\ 4 \end{bmatrix}$ have a sol'n?

Long method for determining IF there is a solution:

$$\left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & 9 & 3 \\ 2 & 5 & 8 & 7 & 22 & 7 \\ 3 & 7 & 12 & 8 & 31 & 8 \\ 1 & 2 & 5 & 4 & 9 & 4 \end{array}\right] \to \left[\begin{array}{cccc|cc} 1 & 2 & 4 & 3 & * & * \\ 0 & 1 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right]$$

Shorter method for determining IF there is a solution WHEN you know a basis for the column space: