Math 2418 Linear Algebra Exam #2
October 30, 2002 SHOW ALL WORK

1.) Let $A = \begin{bmatrix} 2 & 4 & 2 & 2 \\ 4 & 8 & 5 & 5 \\ 2 & 4 & 3 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & 2 & 2 & 1 & 0 \\ 4 & 8 & 5 & 5 & 0 & 0 \\ 2 & 4 & 3 & 3 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[5] 1a.) A basis for the column space of A is

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

[3] 1b.) Find an equation for the hyperplane determined by the column space of A.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

[2] 1c.) Rank $A = \underline{2}$

[5] 1d.) A basis for the row space of A is

$[2 \quad 4 \quad 2 \quad 2], [0 \quad 0 \quad 1 \quad 1]$

[5] 1e.) A basis for the nullspace of A is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Solve $A\mathbf{x} = \mathbf{b}$ for the following values of $\mathbf{b}$:

[2] 1f.) when $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\quad \mathbf{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

[1] 1g.) when $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\quad \mathbf{x} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ -2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

[1] 1h.) when $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\quad \mathbf{x} = \underline{\textit{no solution}}$

[3] 1i.) Find a basis for R^3 which includes your basis vectors for the column space of A in problem 1a.

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[3] 1j.) Write $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ as a linear combination of the basis vectors in problem 1i.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[2] 1k.)The coordinate vector of $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to the basis vectors in problem 1i is

$$\begin{bmatrix} \frac{5}{2} \\ -2 \\ 0 \end{bmatrix}$$

[1] 1l.) A different basis for R^3 which includes your basis vectors for the column space of A is

$$\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

[1] 1m.) Write $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors in problem 1l.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

[1] 1m.) The coordinate vector of $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ with respect to the basis vectors in problem 1l is

$$\begin{bmatrix} \frac{5}{2} \\ -2 \\ 0 \end{bmatrix}$$

[4] 1n.) If $T(\mathbf{x}) = A\mathbf{x}$, then domain of $T = \underline{R^4}$ and codomain of $T = \underline{R^3}$.

[3] 1o.) Is T one-to-one? <u>*no*</u>

[3] 1p.) Is T onto? <u>*no*</u>

[5] 2a.) Express $3 - 2t$ as a linear combination of $1 + 2t$ and $2 + 8t$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \end{bmatrix} \rightarrow$$

Answer 2a.) $\underline{3 - 2t = 7(1 + 2t) - 2(2 + 8t)}$.

[2] 2b.) The coordinate vector of $2 + 5t$ relative to the basis $\{1 + 2t, 3 + 8t\}$ is $\underline{\begin{bmatrix} 7 \\ -2 \end{bmatrix}}$.

[2] 2c.) The coordinate vector of $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ relative to the basis $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \end{bmatrix}\}$ is $\underline{\begin{bmatrix} 7 \\ -2 \end{bmatrix}}$.

[2] 3a.) Is $\{1 + t^2,\ 3 - 4t + 5t^2\}$ a basis for P_2? *no*.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$, thus too few vectors (does not span).

[2] 3b.) Is $\{1 + t^2,\ 3 - 4t + 5t^2,\ 3 - 5t - 9t^2,\ 5 - 2t + t^2\}$ a basis for P_2? *no*.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ -9 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$,thus too many vectors (not linearly independent).

[2] 3c.) Is $\{1 + t^2,\ 2 + 2t^2,\ 3 + 3t^2,\ 3 - 4t + 5t^2\}$ a basis for P_2? *no*.

$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$, thus too many vectors (not linearly independent). Alternatively too few of the right vectors (span of first 3 vectors is a 1-dimensional line, thus this set of 4 vectors does not span a 3-dimensional space. Thus this is an example where there are both too many vectors and too few vectors.

[2] 3d.) Is $\{1,\ 3 - 4t,\ 2 - 9t + 5t^2\}$ a basis for P_2? *yes*.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -9 \\ 5 \end{bmatrix}$, thus exactly the right number and right kind of vectors (linearly independent and spans).

[7] 4.) Determine if $T(x) = 3x + 2$ is linear. Prove your answer.

$T(0) = 2 \neq 0$. Thus, T is not linear.

Alternate proof:

$$T(1) + T(1) = 5 + 5 = 10,$$

$$T(1 + 1) = T(2) = 8.$$

$$\text{Thus, } T(1) + T(1) \neq T(1 + 1) \text{ and hence } T \text{ is not linear.}$$

2nd Alternate proof:

$$2T(1) = 2(5) = 10,$$

$$T(2(1)) = T(2) = 8.$$

$$\text{Thus, } 2T(1) \neq T(2(1)) \text{ and hence } T \text{ is not linear.}$$

Note you only need to show that some part of the definition of linear or a consequence of this definition (i.e. T(0) = 0) fails for one example. Note also, I do want a specific example with real numbers. In more complicated situations, people have tried to show a definition fails by showing that it fails for a general case. However, sometimes it turns out that the general case can't happen and is thus not a counter-example. A real example is safer than a general example.

[8] 5.) Show that the set of all polynomials of the form $a_1x + a_3x^3$ is a subspace of P_3.

$a_1x + a_3x^3 + b_1x + b_3x^3 = (a_1 + b_1)x + (a_3 + b_3)x^3$

$k(a_1x + a_3x^3) = ka_1x + ka_3x^3$

Thus, the set of all polynomials of the form $a_1x + a_3x^3$ is a subspace of P_3.

(Why this proof works: We need to show this set is closed under addition. Thus we take two arbitrary elements from this set. An arbitrary element looks like constant$_1$ times x plus constant$_2$ times x^3. Thus this set is closed under addition since the sum of two arbitrary elements is a polynomial of the form constant$_1$ times x plus constant$_2$ times x^3 (where constant$_1 = a_1 + b_1$ and constant$_2 = a_3 + b_3$. Similarly, this set is also closed under scalar multiplication.)

[7] 6.) Find the standard matrix for the following composition of linear operators on R^2:
A rotation of 90^o, followed by a reflection about the x-axis.

A rotation of 90^o: $\begin{bmatrix} cos90^o & -sin90^o \\ sin90^o & cos90^o \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

A reflection about the x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

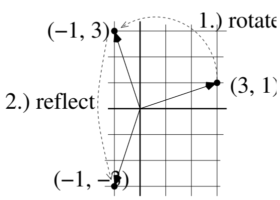
Thus, a (counter-clockwise) rotation of 90^o, followed by a reflection about the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Check: $\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{rotation\ of\ 90^o} \begin{bmatrix} y \\ -x \end{bmatrix} \xrightarrow{reflection\ about\ the\ x-axis} \begin{bmatrix} -y \\ -x \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$$

For example:



Answer$=\underline{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}$

[4] 7a.) The projection of $\begin{bmatrix} 10 \\ 2 \\ 5 \end{bmatrix}$ on $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \underline{\begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}}$

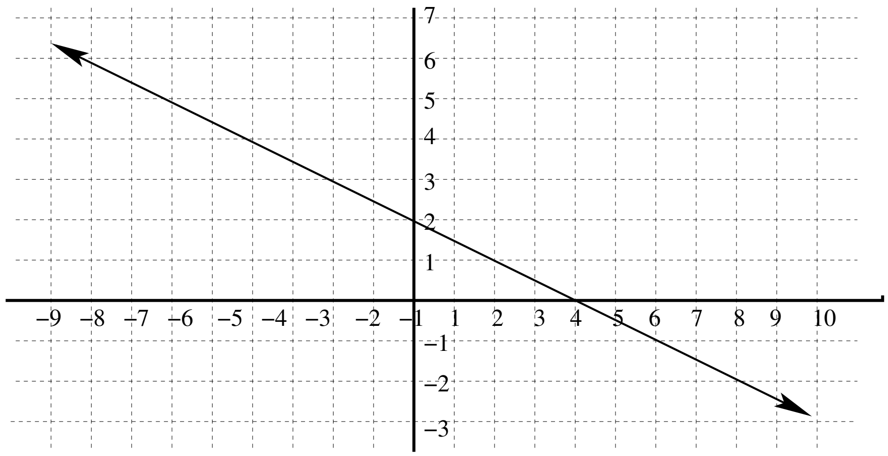
$\begin{bmatrix} 10 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 10 + 10 = 20$ $\qquad$ $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 + 4 = 5$

$$\frac{20}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix}$$

[4] 7b.) The vector component of $\begin{bmatrix} 10 \\ 2 \\ 5 \end{bmatrix}$ orthogonal to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \underline{\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}}$

$$\begin{bmatrix} 10 \\ 2 \\ 5 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$$

8.) Find the equation of the following line:



[4] 8a.) in point-parallel vector form: $\underline{(x, y) = (0, 2) + t(2, -1)}$

[3] 8b.) in point-normal vector form: $\underline{(1, 2) \cdot [(x, y) - (0, 2)] = 0}$

9.) Circle T for true and F for False.

[2] 9a.) If A is a 3×5 matrix, then the columns of A must be linearly dependent.

T

[2] 9b.) If A is a 3×3 matrix, then the columns of A must be a basis.

F